# Optimal Tariffs with Foreign Tariff Revenue

An exact multisector sufficient-statistics formula

**Abstract**

We derive unilateral optimal tariffs in a multisector gravity economy in which every country rebates its tariff revenue to households. Foreign tariff rates are fixed, but foreign tariff revenue and expenditure respond endogenously. At every finite interior local optimum, home tariffs remain uniform across active sectors within each exporter. The updated formula isolates each exporter's own tariff and uses only equilibrium expenditure shares, sales shares, income shares, and sectoral trade elasticities. A new share-based moment captures foreign income feedback. Its contribution affects both the cross-exporter coefficients and the own-tariff denominator, which remains strictly positive. General positive-tariff and global-existence claims are not needed for the result; in fact, nonnegative fixed foreign tariffs can support a strictly locally optimal home import subsidy. The proof is checked by unrestricted numerical optimization, derivative and limiting-case audits, 85-digit recalculation, and an exact rational example.

## 1 Model

There are $N \geq 2$ countries, $K \geq 1$ active sectors, and a home country $i$. Let $\mathcal{F} = \{1, \ldots, N\} \setminus \{i\}$. Origin countries are indexed by $n$ or $j$, destinations by $d$ or $m$, and sectors by $k$. Country $n$ has an inelastic labor endowment $L_n > 0$, wage $w_n > 0$, and labor income $I_n \equiv w_n L_n$. Labor is the only factor, is mobile across sectors within a country, and is internationally immobile. There are no intermediate inputs, international transfers, or trade deficits.

Preferences are Cobb–Douglas across sectors, with common weights $\beta_k > 0$ satisfying $\sum_k \beta_k = 1$. The sectoral gravity elasticity is $\epsilon_k > 0$. Positive finite constants $\bar{\chi}_{nd,k}$ collect technology and iceberg costs. The reduced-form demand and price equations below define the model; no additional microfoundation is needed for the derivation.

### 1.1 Tariffs, demand, and prices

Write $\tau_{nd,k} = 1 + t_{nd,k} > 0$ for a gross tariff factor, with $\tau_{nn,k} = 1$. Home chooses every $\tau_{ni,k}$ for $n \in \mathcal{F}$ and every active $k$. At every foreign destination $d \in \mathcal{F}$, the complete array $\tau_{nd,k} = \bar{\tau}_{nd,k} > 0$ is fixed. It may vary by origin, destination, and sector. In particular, foreign import tariffs can apply to home's exports and to trade among foreign countries. There is no retaliation: $\mathrm{d}\bar{\tau}_{nd,k} = 0$. The usual case $\bar{\tau}_{nd,k} \geq 1$ is included; allowing positive gross factors below one does not change the proof.

The tariff-inclusive expenditure shares and sectoral price indices are

$$\lambda_{nd,k} = \frac{\bar{\chi}_{nd,k}(\tau_{nd,k} w_n)^{-\epsilon_k}}{\sum_a \bar{\chi}_{ad,k}(\tau_{ad,k} w_a)^{-\epsilon_k}}, \qquad \sum_n \lambda_{nd,k} = 1, \tag{1}$$

$$P_{d,k} = \left[\sum_a \bar{\chi}_{ad,k}(\tau_{ad,k} w_a)^{-\epsilon_k}\right]^{-1/\epsilon_k}, \qquad P_d = \prod_k P_{d,k}^{\beta_k}. \tag{2}$$

Country $d$ spends $\beta_k Y_d$ in sector $k$. The f.o.b. value of the corresponding origin–destination flow, including domestic sales, is

$$X_{nd,k} = \frac{\beta_k \lambda_{nd,k} Y_d}{\tau_{nd,k}}. \tag{3}$$

The factor $\tau_{nd,k}^{-1}$ belongs to *every* tariff-bearing flow, including flows into foreign destinations.

### 1.2 Income and equilibrium

Every government rebates tariff revenue lump sum to its own households:

$$R_d = \sum_{n,k} (\tau_{nd,k} - 1) X_{nd,k}, \qquad Y_d = I_d + R_d \quad \text{for every } d. \tag{4}$$

Define $\rho_d \equiv I_d / Y_d$. Using (3) in (4) gives the exact revenue identity

$$\boxed{\rho_d = \sum_{n,k} \frac{\beta_k \lambda_{nd,k}}{\tau_{nd,k}}, \qquad Y_d = \frac{I_d}{\rho_d}, \qquad \sum_{n,k} X_{nd,k} = I_d.} \tag{5}$$

Thus f.o.b. absorption equals labor income in every destination. Producer clearing requires

$$I_n = \sum_{d,k} X_{nd,k} \qquad (n = 1, \ldots, N). \tag{6}$$

Together, (5) and (6) imply balanced international trade in f.o.b. values. One clearing equation is redundant. Fix a wage numeraire and solve the remaining equations.

Home maximizes $W_i = Y_i / P_i$ over its unrestricted tariff array, subject to these equilibrium conditions. The theorem concerns finite interior local maxima of this policy–equilibrium problem, and hence any finite interior global maximum. Equivalently, it applies along a locally regular selected equilibrium branch. No uniqueness of the global wage equilibrium is assumed. The original domain $t_{ni,k} > -1$ permits import subsidies. A policy problem with additional constraints $t_{ni,k} \geq 0$ has boundary conditions in place of some of the equalities below.

If a sector has $\beta_k = 0$, its flows and welfare weight are zero and its tariffs are unidentified. Such sectors must be deleted before applying the theorem; no uniformity claim is made for them.

## 2 Shares and the updated theorem

### 2.1 The original sufficient statistics still suffice

Retain the expenditure, sales, and disposable-income shares

$$\lambda_{nd} = \sum_k \beta_k \lambda_{nd,k}, \qquad r_{nd,k} = \frac{X_{nd,k}}{I_n}, \qquad r_{nd} = \sum_k r_{nd,k}, \qquad y_n = \frac{Y_n}{\sum_a Y_a}. \tag{7}$$

Producer clearing gives $\sum_d r_{nd} = 1$. Since domestic tariffs are zero,

$$r_{nn} = \frac{\lambda_{nn} Y_n}{I_n} = \frac{\lambda_{nn}}{\rho_n}. \tag{8}$$

Consequently, the following quantities require no new observations:

$$\boxed{\rho_n = \frac{\lambda_{nn}}{r_{nn}}, \qquad \omega_n \equiv \rho_n y_n, \qquad \frac{\omega_n}{\omega_d} = \frac{I_n}{I_d}.} \tag{9}$$

The $\omega_n$ need not sum to one; only their ratios are used.

For clarity, introduce the destination-normalized f.o.b. absorption shares

$$q_{nd,k} \equiv \frac{\omega_n}{\omega_d} r_{nd,k} = \frac{X_{nd,k}}{I_d}, \qquad q_{nd} = \sum_k q_{nd,k}, \qquad Q_{d,k} = \sum_a q_{ad,k}. \tag{10}$$

They satisfy $\sum_n q_{nd} = \sum_k Q_{d,k} = 1$. The central new moment is, for any origin $n$ and any foreign destination $d \in \mathcal{F}$,

$$\boxed{\zeta_{nd} \equiv \sum_k \epsilon_k \left(Q_{d,k}\lambda_{nd,k} - q_{nd,k}\right).} \tag{11}$$

This is defined entirely by shares and structural elasticities. The proof shows that it is the elasticity of $\rho_d$ with respect to origin $n$'s wage, holding all foreign tariff rates fixed. Thus it captures the additional foreign-income response generated by changing tariff revenue.

## 2.2 Coefficients and own-tariff isolation

For $n, j \in \mathcal{F}$, define the cross-exporter coefficient

$$\boxed{C_{nj} \equiv \frac{\omega_j}{\omega_n}\left[r_{jn} + \sum_{d\in\mathcal{F}}\sum_k \epsilon_k r_{jd,k}\lambda_{nd,k} - \sum_{d\in\mathcal{F}} r_{jd}\zeta_{nd}\right].} \tag{12}$$

The first index $n$ identifies the exporter whose home tariff is being determined; the second $j$ identifies the other tariff multiplying the coefficient. Define

$$\boxed{D_n \equiv \sum_{d\in\mathcal{F}\setminus\{n\}} r_{nd} + \sum_{d\in\mathcal{F}}\sum_k \epsilon_k r_{nd,k}(1-\lambda_{nd,k}) + \sum_{d\in\mathcal{F}} r_{nd}\zeta_{nd}.} \tag{13}$$

All objects in (12)–(13) are explicit functions of $(\lambda_{nd,k}, \lambda_{nd}, r_{nd,k}, r_{nd}, y_n, \epsilon_k)$.

**Theorem 1** (Optimal tariffs with fixed foreign tariff rates)**.** *Assume the model in Section 1, with strictly positive active sector weights, trade elasticities, labor endowments, and gravity constants. At every finite interior local maximizer of the unrestricted $(N-1)K$-instrument policy–equilibrium problem, tariffs are uniform across active sectors within an exporter:*

$$t^*_{ni,k} = t^*_{ni,g} \equiv t^*_{ni} \qquad (n \in \mathcal{F};\ k, g = 1, \ldots, K). \tag{14}$$

*For each foreign exporter $n$, the exact own-tariff-isolated formula is*

$$\boxed{1 + t^*_{ni} = 1 + \frac{r^*_{ni} + \sum_{j\in\mathcal{F}\setminus\{n\}}\left(\dfrac{\lambda^*_{ji}y^*_i}{\rho^*_j r^*_{ji} y^*_j} - 1\right) C^*_{nj}}{D^*_n}.} \tag{15}$$

*Here $C_{nj}$, $D_n$, and $\zeta_{nd}$ are given by (12), (13), and (11), and $\rho_j = \lambda_{jj}/r_{jj}$. The denominator is strictly positive. The right-hand side contains no reconstruction of exporter $n$'s own tariff, no foreign tariff rate, no primitive, no wage derivative, no multiplier, and no matrix inverse.*

Equation (15) has the same structure as the no-foreign-tariff formula: own exposure $r_{ni}$ plus other partners' tariffs reconstructed from shares, divided by an explicit own coefficient. The new income moment enters $C_{nj}$ with a minus sign and $D_n$ with a plus sign. All starred statistics are evaluated at the optimum. As in other exact sufficient-statistic formulas, equilibrium shares themselves depend on policy; the display is not a claim that one can insert arbitrary initial shares to obtain an optimal counterfactual without solving equilibrium.

*Remark* 1 (What is and is not asserted). The theorem supplies necessary conditions at every finite interior optimum. Conversely, sectoral uniformity together with (15) characterizes interior constrained stationarity, as proved below. Stationarity alone is not sufficient for a maximum. We make no unconditional claim of global attainment or equilibrium uniqueness under general fixed foreign tariff arrays. Nor must every optimal net tariff be positive: the exact example in Appendix A is a strict local maximum with a home import subsidy, despite nonnegative foreign tariffs.

# 3 Proof of Theorem 1

## 3.1 Equilibrium coordinates and home revenue

For the proof only, normalize by home labor income and set

$$m_n = \frac{I_n}{I_i}, \quad x_n = \log \frac{w_n}{w_i}, \quad c_{n,k} = x_n + \log \tau_{ni,k}, \quad b_{n,k} = \frac{X_{ni,k}}{I_i}, \quad b_n = \sum_k b_{n,k}, \quad B_k = \sum_n b_{n,k}. \tag{16}$$

Here $x_i = c_{i,k} = 0$ and $m_i = 1$. Home shares depend only on $c$; foreign shares and their $q$-shares depend only on $x$, because foreign rates are fixed. Conversely, every finite $(c, x)$ implements positive home gross tariffs $\tau_{ni,k} = \exp(c_{n,k} - x_n)$. We have

$$b_{n,k} = \frac{\beta_k \lambda_{ni,k}}{\rho_i \tau_{ni,k}}, \qquad \sum_n b_n = \sum_k B_k = 1. \tag{17}$$

The foreign producer-clearing equations become

$$H_j(c, x) \equiv m_j - b_j - \sum_{d \in \mathcal{F}} m_d q_{jd}(x) = 0, \qquad j \in \mathcal{F}. \tag{18}$$

Their sum implies the omitted home equation by f.o.b. absorption adding up. Apart from the constant $\log L_i$, home log welfare is

$$V(c, x) = -\log \rho_i - \sum_k \beta_k \log(P_{i,k}/w_i). \tag{19}$$

The term $-\log \rho_i$ retains the complete home tariff-revenue feedback.

## 3.2 Unrestricted instrument derivatives and constraint rank

For a foreign origin $a$, differentiating with respect to $c_{a,k}$ at fixed $x$ gives

$$\partial_{c_{a,k}} \lambda_{ni,g} = -\mathbf{1}\{g = k\} \epsilon_k \lambda_{ni,k} (\mathbf{1}\{n = a\} - \lambda_{ai,k}), \tag{20}$$

$$\partial_{c_{a,k}} \log \rho_i = \epsilon_k \lambda_{ai,k} B_k - (1 + \epsilon_k) b_{a,k}, \tag{21}$$

$$V_{c_{a,k}} = (1 + \epsilon_k) b_{a,k} - \lambda_{ai,k} (\beta_k + \epsilon_k B_k), \tag{22}$$

$$H_{j,c_{a,k}} = (1 + \epsilon_k)(\mathbf{1}\{j = a\} b_{j,k} - b_j b_{a,k}) - \epsilon_k \lambda_{ai,k} (b_{j,k} - b_j B_k). \tag{23}$$

Foreign revenue introduces no new term in this block: every foreign flow is fixed when $c$ varies at fixed wages.

We verify full row rank rather than assuming it. Fix an active $k$, put $e = \epsilon_k$, and suppose the square block $(H_{j,c_{a,k}})_{j,a\in\mathcal{F}}$ annihilates a vector $v$. Define

$$u_j = \frac{b_{j,k}}{b_j} > 0, \qquad s = \sum_{a\in\mathcal{F}} b_{a,k} v_a, \qquad \ell = \sum_{a\in\mathcal{F}} \lambda_{ai,k} v_a, \qquad \gamma = (1+e)s - eB_k\ell.$$

Dividing row $j$ by $b_j$ in (23) yields

$$v_j = \frac{\gamma}{(1+e)u_j} + \frac{e\ell}{1+e}. \tag{24}$$

Multiplying by $b_{j,k}$ and summing over foreign $j$ gives $\gamma b_i + e b_{i,k}\ell = 0$. Multiplying instead by $\lambda_{ji,k}$ gives

$$\ell = \frac{\gamma \sum_{j\in\mathcal{F}} \lambda_{ji,k}/u_j}{1 + e\lambda_{ii,k}}.$$

Substitution into the first identity forces $\gamma = 0$, because $b_i > 0$ and every term in the remaining bracket is positive. Then $\ell = s = 0$ and (24) gives $v = 0$. Thus each active-sector instrument block is nonsingular, and the equality-constraint Jacobian has full row rank everywhere in the finite interior. Ordinary constrained first-order conditions are therefore necessary.

### 3.3 Sectoral uniformity from all instruments

Use the Lagrangian $\mathcal{L} = V + \sum_{j\in\mathcal{F}} \mu_j H_j$ and write

$$\mu_i = 0, \qquad M = \sum_{j\in\mathcal{F}} \mu_j b_j, \qquad v_n = 1 + \mu_n - M, \qquad S_k = \sum_n v_n b_{n,k}. \tag{25}$$

Adding up implies $\sum_k S_k = \sum_n v_n b_n = 1$. Equations (22) and (23) make every unrestricted instrument condition exactly

$$(1+\epsilon_k) b_{a,k} v_a = \lambda_{ai,k}(\beta_k + \epsilon_k S_k), \qquad a \in \mathcal{F}. \tag{26}$$

Summing (26) over foreign origins within sector $k$ gives

$$(1 + \epsilon_k\lambda_{ii,k}) S_k = \beta_k(1 - \lambda_{ii,k}) + (1+\epsilon_k)(1-M) b_{i,k}. \tag{27}$$

Let $z = (1-M)/\rho_i$, substitute $b_{i,k} = \beta_k\lambda_{ii,k}/\rho_i$, and sum over $k$. The identity $\sum_k S_k = 1$ becomes

$$0 = (z-1) \sum_k \frac{\beta_k(1+\epsilon_k)\lambda_{ii,k}}{1+\epsilon_k\lambda_{ii,k}}. \tag{28}$$

The sum is strictly positive; hence $z = 1$, $M = 1 - \rho_i$, and (27) gives $S_k = \beta_k$. The unrestricted conditions reduce to $b_{a,k}(\rho_i + \mu_a) = \beta_k\lambda_{ai,k}$. Using (17),

$$\boxed{\tau_{ai,k} = 1 + \frac{\mu_a}{\rho_i}, \qquad \mu_a = \rho_i t_{ai}.} \tag{29}$$

This proves sectoral uniformity without imposing it on the government's instrument set. No division by an expression of unknown sign was used.

### 3.4 The endogenous foreign-income derivative

Hold foreign tariff rates fixed. For $d \in \mathcal{F}$ and $a \in \mathcal{F}$,

$$\partial_{x_a} \lambda_{jd,k} = -\epsilon_k \lambda_{jd,k} (\mathbf{1}\{j = a\} - \lambda_{ad,k}). \tag{30}$$

Since $\beta_k \lambda_{jd,k} / (\bar{\tau}_{jd,k} \rho_d) = q_{jd,k}$, differentiate (5) to obtain

$$\partial_{x_a} \log \rho_d = \sum_k \epsilon_k \left[ \lambda_{ad,k} \sum_j q_{jd,k} - q_{ad,k} \right] = \zeta_{ad}, \tag{31}$$

$$\partial_{x_a} \log Y_d = \mathbf{1}\{a = d\} - \zeta_{ad}, \tag{32}$$

$$\partial_{x_a} R_d = \mathbf{1}\{a = d\} R_d - Y_d \zeta_{ad}. \tag{33}$$

Equation (33) explicitly exhibits changing revenue at fixed rates. Holding foreign revenue fixed, or treating foreign income as proportional to its wage, would omit this effect. The required f.o.b. share derivative is

$$\partial_{x_a} q_{jd,k} = q_{jd,k} \left[ -\epsilon_k \mathbf{1}\{j = a\} + \epsilon_k \lambda_{ad,k} - \zeta_{ad} \right]. \tag{34}$$

As a homogeneity check, extend $\zeta_{ad}$ to every origin by its share definition. Then $\sum_a \zeta_{ad} = 0$, including the home-origin term.

### 3.5 Wage conditions and the corrected coefficients

At fixed $c$, home expenditure shares and prices relative to $w_i$ are fixed. Direct differentiation of (17)–(19) gives

$$V_{x_a} = -b_a, \qquad \partial_{x_a} b_j = b_j (\mathbf{1}\{j = a\} - b_a), \qquad \partial_{x_a} m_j = \mathbf{1}\{j = a\} m_j. \tag{35}$$

Differentiate (18), use (34), and substitute equilibrium clearing $m_j - b_j = \sum_{d \in \mathcal{F}} m_d q_{jd}$. The complete wage block is

$$\begin{aligned} H_{j,x_a} = \mathbf{1}\{j = a\} \sum_{d \in \mathcal{F}, k} m_d (1 + \epsilon_k) q_{jd,k} + b_j b_a - m_a q_{ja} \\ - \sum_{d \in \mathcal{F}, k} m_d \epsilon_k q_{jd,k} \lambda_{ad,k} + \sum_{d \in \mathcal{F}} m_d q_{jd} \zeta_{ad}. \end{aligned} \tag{36}$$

The last term is the foreign revenue correction. The wage conditions are $-b_a + \rho_i \sum_{j \in \mathcal{F}} t_{ji} H_{j,x_a} = 0$. Home's revenue identity and uniformity give

$$\sum_{j \in \mathcal{F}} t_{ji} b_j = \frac{1 - \rho_i}{\rho_i}. \tag{37}$$

After inserting (36), the $b_j b_a$ terms cancel the corresponding home revenue term. The resulting equality, before normalization, is

$$\begin{aligned} b_a = t_{ai} \sum_{d \in \mathcal{F}, k} m_d (1 + \epsilon_k) q_{ad,k} \\ - \sum_{j \in \mathcal{F}} t_{ji} \left[ m_a q_{ja} + \sum_{d \in \mathcal{F}, k} m_d \epsilon_k q_{jd,k} \lambda_{ad,k} - \sum_{d \in \mathcal{F}} m_d q_{jd} \zeta_{ad} \right]. \end{aligned} \tag{38}$$

Divide by $m_a$, use $b_a/m_a = r_{ai}$ and $m_d q_{jd,k}/m_a = (\omega_j/\omega_a) r_{jd,k}$, and define

$$A_a \equiv \sum_{d\in\mathcal{F},k} (1+\epsilon_k) r_{ad,k} = 1 - r_{ai} + \sum_{d\in\mathcal{F},k} \epsilon_k r_{ad,k}. \tag{39}$$

We obtain the exact wage-condition system

$$\boxed{A_a t_{ai} = r_{ai} + \sum_{j\in\mathcal{F}} C_{aj} t_{ji}.} \tag{40}$$

In particular, the foreign coefficient is (12); it is not obtained by mechanically retaining the old expenditure-share weights. With discriminatory foreign tariffs the correct identity is

$$X_{jd,k}\lambda_{ad,k} = \frac{\overline{\tau}_{ad,k}}{\overline{\tau}_{jd,k}} X_{ad,k}\lambda_{jd,k}, \qquad d \in \mathcal{F}. \tag{41}$$

The tariff-free transpose identity generally fails. Keeping the sales-share orientation in (12) avoids introducing any unobserved tariff ratio.

### 3.6 Own-tariff isolation and denominator positivity

Moving the $j = a$ term in (40) to the left gives

$$D_a t_{ai} = r_{ai} + \sum_{j\in\mathcal{F}\setminus\{a\}} C_{aj} t_{ji}, \qquad D_a = A_a - C_{aa}. \tag{42}$$

Expanding $A_a - C_{aa}$ gives precisely (13). To prove its strict positivity even when $\zeta_{ad}$ is negative, fix a foreign destination $d$. Use $r_{ad,k} = (\omega_d/\omega_a) q_{ad,k}$ and expand (11):

$$\begin{aligned} &\sum_k \epsilon_k r_{ad,k}(1-\lambda_{ad,k}) + r_{ad}\zeta_{ad} \\ &\quad = \frac{\omega_d}{\omega_a} \sum_k \epsilon_k \left[(1-q_{ad})(1-\lambda_{ad,k}) q_{ad,k} + q_{ad}\lambda_{ad,k}(Q_{d,k} - q_{ad,k})\right] > 0. \end{aligned} \tag{43}$$

Positive primitives and active sectors imply $0 < q_{ad} < 1$, $0 < \lambda_{ad,k} < 1$, $q_{ad,k} > 0$, and $Q_{d,k} - q_{ad,k} > 0$. Every term in the last expression is positive. Adding $\sum_{d\in\mathcal{F}\setminus\{a\}} r_{ad} \geq 0$ proves $D_a > 0$.

Finally, uniformity implies

$$r_{ji} = \frac{\lambda_{ji} Y_i}{\tau_{ji} I_j}, \qquad t_{ji} = \frac{\lambda_{ji} y_i}{\rho_j r_{ji} y_j} - 1. \tag{44}$$

Substitute (44) for $j \neq a$ in (42). This is (15), with the own tariff isolated. All auxiliary quantities are reconstructed from the original shares through (9)–(11); they are numeraire invariant.

For the converse stationarity statement, take a finite equilibrium satisfying uniformity and (15), and set $\mu_j = \rho_i t_{ji}$. Equation (37) gives $M = 1 - \rho_i$, and hence $S_k = \sum_n \rho_i \tau_{ni} b_{n,k} = \beta_k$. Therefore every unrestricted condition (26) holds. The formula gives (40), which is equivalent to the wage conditions already derived. Full row rank then establishes constrained stationarity. This completes the proof. □

# 4 Reductions and interpretation

## 4.1 Zero foreign tariffs: exact recovery of the original formula

If every foreign tariff is zero, $\rho_d = 1$, $q_{nd,k} = \beta_k \lambda_{nd,k}$, $Q_{d,k} = \beta_k$, and $\zeta_{nd} = 0$ for $d \in \mathcal{F}$. The tariff-free flow identity gives

$$C_{nj} = \lambda_{jn} + \sum_{d \in \mathcal{F},k} \epsilon_k r_{nd,k} \lambda_{jd,k}. \tag{45}$$

Thus (15) reduces exactly to

$$1 + t_{ni} = 1 + \frac{r_{ni} + \sum_{j \in \mathcal{F}\setminus\{n\}} \left( \frac{\lambda_{ji} y_i}{r_{ji} y_j} - 1 \right) \left[ \lambda_{jn} + \sum_{d \in \mathcal{F},k} \epsilon_k r_{nd,k} \lambda_{jd,k} \right]}{\sum_{d \in \mathcal{F}\setminus\{n\}} r_{nd} + \sum_{d \in \mathcal{F},k} \epsilon_k r_{nd,k} (1 - \lambda_{nd,k})}. \tag{46}$$

Optimal-share stars are suppressed in this section for readability.

## 4.2 Two countries and one sector

With two countries, there is only one foreign exporter and the cross-exporter sum is empty. Therefore

$$t_{ni} = \frac{r_{ni}}{D_n} > 0 \qquad (N = 2). \tag{47}$$

This remains true with arbitrary fixed foreign sectoral tariffs.

With one sector, $Q_{d,1} = 1$ and $\zeta_{nd} = \epsilon(\lambda_{nd} - q_{nd})$. The coefficients simplify to

$$C_{nj} = q_{jn} + \epsilon \sum_{d \in \mathcal{F}} r_{nd} q_{jd}, \tag{48}$$

$$D_n = \sum_{d \in \mathcal{F}\setminus\{n\}} r_{nd} + \epsilon \sum_{d \in \mathcal{F}} r_{nd} (1 - q_{nd}). \tag{49}$$

This has the familiar single-sector structure with foreign f.o.b. absorption shares $q$ in place of foreign tariff-inclusive expenditure shares. All coefficients $C_{nj}$ are positive in this case.

For completeness, extend (12) to $j = i$ for this argument. Summing over all origins gives

$$\sum_{j=1}^{N} C_{nj} = 1 + \sum_{d \in \mathcal{F},k} \epsilon_k r_{nd,k} = A_n + r_{ni}. \tag{50}$$

With $\tau_{ii} = 1$, the wage condition becomes $A_n \tau_{ni} = \sum_j C_{nj} \tau_{ji}$. In the one-sector case, if a foreign gross tariff were at most one, choose a foreign origin $n$ minimizing all gross home tariffs, including the domestic factor one. Positivity of $C_{nj}$ would imply $A_n \tau_{ni} \geq \tau_{ni}(A_n + r_{ni})$, a contradiction. Hence all stationary net tariffs are positive when $K = 1$. A negative-tariff local optimum in this model therefore requires at least three countries and at least two active sectors.

## 4.3 Why fixed foreign tariffs change more than one term

There are three linked corrections. First, another partner's home tariff is reconstructed with its labor-income share $\rho_j$, as in (44). Second, discriminatory foreign wedges change the relation

between expenditure and producer sales, so the competition coefficient must respect (41). Third, foreign expenditure responds through $-\zeta_{nd}$ in (32). Absorbing foreign tariff factors into the gravity constants reproduces expenditure shares at a point, but does not remove the f.o.b. wedge or its income consequences.

The income effect can change signs across countries. Consequently, $C_{nj}$ can be negative even when all foreign tariff rates are nonnegative. The positive-denominator result survives because its own-origin terms combine as in (43); the old positive-weight argument for every home net tariff does not survive in general. Appendix A makes this distinction concrete.

# 5 Numerical and exact-arithmetic verification

## 5.1 Independent primitive-based optimization

The main audit evaluates the primitive gravity equations, solves every country's revenue identity, and maximizes $\log(Y_i/P_i)$ over all $(N-1)K$ log gross home tariffs. It does not use (15) to choose or solve for an optimum. Wage derivatives come from implicit differentiation of producer clearing. The formula is evaluated only after optimizing and refining full-policy stationarity.

There are 150 principal randomized economies, with $N \in \{2,3,4,5\}$ and $K \in \{1,2,3,4\}$; asymmetric labor endowments and gravity constants; positive sector weights; and elasticities drawn between 0.5 and 9. Most foreign tariffs are nonnegative. Thirty cases also permit foreign subsidies, checking the theorem's full positive-gross-factor domain. The first 20 economies are each reoptimized from zero home tariffs and from an independent nonuniform vector, for 40 additional full-policy runs. Forty additional two-country economies are optimized to check (47).

For the 150 principal runs, Table 1 reports maximum errors. The forward tariff error is scaled as $|a-b|/(1+|a|+|b|)$. Derivative and identity comparisons use the same scaling when stated as scaled errors; clearing, gradient, and formula equation residuals are unscaled maximum absolute residuals. Producer clearing is measured as $\sum_{d,k} X_{nd,k}/I_n - 1$.

Table 1: Independent double-precision checks: 150 unrestricted optimizations

| Diagnostic | Maximum error |
|---|---|
| Producer clearing, including the omitted country | $1.33 \times 10^{-15}$ |
| Full unrestricted welfare gradient | $1.67 \times 10^{-16}$ |
| Within-exporter log gross tariff range | $7.22 \times 10^{-16}$ |
| Formula equation residual | $3.65 \times 10^{-15}$ |
| Scaled forward tariff error | $3.52 \times 10^{-14}$ |
| Tariff reconstruction from shares | $5.23 \times 10^{-16}$ |
| Full-policy Hessian asymmetry | $5.00 \times 10^{-16}$ |
| Multistart log-welfare difference | $1.11 \times 10^{-15}$ |

Every full-policy Hessian is negative definite; the largest of their maximum eigenvalues is $-2.84 \times 10^{-4}$. No economy is discarded. Solver flags are preserved: 65 of 150 initial BFGS calls reported precision loss and five subsequent root calls reported non-success. All 150 points were then refined with an explicit full-system Newton calculation, and every final point satisfied the reported clearing, gradient, curvature, and formula checks. An optimizer's success flag is not used as an optimality certificate.

### 5.2 Derivative, accounting, and limiting-case checks

At 300 separate parameter and policy draws, analytic derivatives of primitive equilibrium and welfare are compared with complex-step derivatives using step $10^{-30}$. The largest scaled discrepancy is $6.72 \times 10^{-16}$. The foreign-income-feedback derivative in (31) agrees within $8.08 \times 10^{-16}$, and the positive-denominator identity (43) within $9.80 \times 10^{-16}$.

An additional 200 adversarial draws use $N = 2, \ldots, 6$, $K = 1, \ldots, 5$, gross tariff factors from 0.001 to 1000, sector weights down to $10^{-6}$, and elasticities from 0.05 to 40. Maximum scaled errors are $1.62 \times 10^{-14}$ for primitive derivatives, $3.96 \times 10^{-15}$ for foreign-income-feedback derivatives, $5.08 \times 10^{-15}$ for the income identity, $2.22 \times 10^{-16}$ for f.o.b. absorption, and $3.33 \times 10^{-15}$ for (43). These are stress checks of identities and derivatives; they are not additional claims of global policy optimization.

Forty separate cases verify each limiting reduction: the no-foreign-tariff coefficients agree within $1.28 \times 10^{-16}$, the one-sector coefficients within $1.96 \times 10^{-16}$, and the two-country optimized scalar formula within $5.24 \times 10^{-15}$. Deliberately deleting the foreign-income correction from the formula produces an equation residual as large as 0.537 in the principal suite. The correction is quantitatively material.

### 5.3 Precision, independent recalculation, and reproducibility

IEEE double precision has machine epsilon approximately $2.22 \times 10^{-16}$. The main equation residuals are at floating-point roundoff scale, allowing for finite sums and conditioning. A small $D_n$ can amplify a tiny equation residual into a larger *forward* tariff error; the $3.52 \times 10^{-14}$ entry in Table 1 is not represented as a one-unit-in-the-last-place guarantee. Extreme subsidy cases also introduce cancellation in income accounting. Numerical checks do not replace the algebraic proof.

To resolve the remaining precision question, a separate implementation uses 85-digit decimal arithmetic and forward automatic differentiation of the *primitive* gravity, revenue, clearing, and welfare expressions. It solves clearing jointly with the full unrestricted welfare gradient, without using the proposed tariff formula in the solver. Five cases include the worst forward-error economy from the double-precision suite, the exact subsidy example, and asymmetric two-, three-, and four-country models. The maximum discrepancies are

$$\begin{aligned} \text{producer clearing} &\leq 1.91 \times 10^{-84}, \\ \text{full policy gradient} &\leq 7.0 \times 10^{-85}, \\ \text{formula forward error} &\leq 1.20 \times 10^{-83}, \\ \text{within-exporter log tariff range} &\leq 1.10 \times 10^{-84}. \end{aligned} \tag{51}$$

An independent 85-digit centered-difference check of the automatic derivatives, using step $10^{-23}$, has error below $5.89 \times 10^{-46}$, consistent with its finite-difference truncation error. It is not labeled a machine-precision finite-difference test.

The companion audit archive contains scripts, deterministic seeds, complete diagnostics, and exact rational certificates. The main command is

```
python audit_independent/audit.py -cases 150 -derivatives 300
  -out audit_independent/results.json
```

The high-precision command is

```
python audit_precision/high_precision.py
```

The exact example is reproduced by

```
python independent_rational_certificate.py
```

and its exact full-policy curvature certificate by

```
python audit_exact_hessian/exact_hessian.py
```

The main audit uses Python 3.12.14, NumPy 2.3.5, and SciPy 1.17.0; the high-precision arithmetic uses Python's standard-library `decimal`, and the exact certificate uses `fractions`. Optimization seeds are 820000–820149, derivative seeds 710000–710299, and stress-test seeds 948000–948199. The zero-foreign-tariff, one-sector, and two-country tests use seeds 935000–935039, 936000–936039, and 937000–937039, respectively. The LaTeX manuscript itself is standalone and needs none of these files to compile.

# A An exact example with a locally optimal import subsidy

This example shows why the theorem must not inherit a universal positive-net-tariff claim for interior local optima. Index countries by $0, 1, 2$, with home 0, and take

$$\beta = (1/2, 1/2), \qquad \epsilon = (1/2, 10), \qquad b = (1/2, 1/4, 1/4), \qquad L = \left(1, \frac{503699}{459443}, \frac{613377}{918886}\right). \tag{52}$$

Normalize all equilibrium wages to one. The foreign expenditure shares and fixed gross tariff factors are listed by origin in Table 2. All foreign net tariffs are nonnegative, and all gross factors are at most ten.

Table 2: Exact foreign data for the three-country, two-sector example

| Destination $d$ | Sector $k$ | $\lambda_{\cdot d,k}$ | $\bar{\tau}_{\cdot d,k}$ |
|---|---|---|---|
| 1 | 1 | $(3/34, 15/17, 1/34)$ | $(2, 1, 10)$ |
| 1 | 2 | $(3/14, 1/14, 5/7)$ | $(1, 1, 10)$ |
| 2 | 1 | $(1/21, 10/21, 10/21)$ | $(10, 10, 1)$ |
| 2 | 2 | $(15/17, 1/34, 3/34)$ | $(2, 5, 1)$ |

Set $t_{00} = 0$ and let the home net tariffs be

$$t_{10} = \frac{69474406339399681}{119758732255434860} = 0.58011975436761365141937389 93\ldots, \tag{53}$$

$$t_{20} = -\frac{151962248441737329}{838311125788044020} = -0.18127189746991261055348900 13\ldots. \tag{54}$$

They are uniform across the two sectors, and both gross factors are positive. Complete the calibration by setting, in each home sector,

$$\rho_0 = \frac{1}{\sum_{n=0}^{2}(1+t_{n0})b_n}, \qquad \lambda_{n0,k} = \rho_0(1+t_{n0})b_n, \qquad \bar{\chi}_{nd,k} = \lambda_{nd,k}\tau_{nd,k}^{\epsilon_k} \tag{55}$$

for every destination, using the fixed foreign factors in the table. These explicit positive primitives reproduce the specified shares at wages one. The labor endowments in (52) solve all producer-clearing equations. All adding-up, revenue, and clearing identities hold exactly in rational arithmetic at this equilibrium.

The isolated tariff equations are the exact rational system

$$\begin{pmatrix} \frac{455016795791}{1301723429272} & -\frac{25907112841}{185960489896} \\ \frac{392885004397}{396291838914} & \frac{438575571791}{396291838914} \end{pmatrix} \begin{pmatrix} t_{10} \\ t_{20} \end{pmatrix} = \begin{pmatrix} \frac{459443}{2014796} \\ \frac{459443}{1226754} \end{pmatrix}. \tag{56}$$

Substituting (53)–(54) gives *exactly zero* residuals, with no rounding tolerance. Both diagonal entries, which are $D_1$ and $D_2$, are positive. The positive lower-left entry means $C_{21} < 0$: this is the sign change that permits the subsidy.

The full four-instrument Hessian of log welfare, allowing wages to adjust and varying each partner–sector log gross tariff separately, has eigenvalues approximately

$$(-1.73563717,\ -0.39299955,\ -0.20655858,\ -0.08991132). \tag{57}$$

There is also an *exact*, rather than floating-point, certificate of strict local maximality. Use two foreign log wage deviations $x$ and four independent home log tariff deviations $u$ around the stated equilibrium. Writing the primitive clearing residuals as $H(x,u)$ and log welfare as $V(x,u)$, compute

$$H_x^{\mathsf{T}}\alpha = V_x, \qquad T = \begin{pmatrix} -H_x^{-1}H_u \\ I_4 \end{pmatrix}, \qquad \mathcal{H} = T^{\mathsf{T}}\left(V'' - \sum_{n=1}^{2}\alpha_n H_n''\right)T. \tag{58}$$

All derivatives at this equilibrium are rational: one can write the gravity weights locally as the baseline rational $\lambda_{nd,k}$ multiplied by $\exp[-\epsilon_k(x_n + u_{nd,k})]$, where $u_{nd,k} = 0$ at foreign destinations. At zero deviations, the exponential and its derivatives with respect to its argument equal one. Log welfare derivatives use only rational ratios of first and second derivatives; its irrational level never enters the calculation.

Second-order automatic differentiation using exact fractions gives a nonsingular $H_x$, zero full constrained gradient, and an exactly symmetric $\mathcal{H}$. If $\Delta_s$ is its $s$th leading principal minor, exact rational comparisons give

$$\begin{aligned} -204/1000 < \Delta_1 < -203/1000, \quad 252/1000 < \Delta_2 < 253/1000, \\ -29/1000 < \Delta_3 < -28/1000, \qquad 12/1000 < \Delta_4 < 13/1000. \end{aligned} \tag{59}$$

These inequalities are exact bounds, not rounded numerical intervals. Sylvester's criterion proves that $\mathcal{H}$ is negative definite. Together with the nonsingular wage Jacobian and exact stationarity, the second-order sufficient conditions prove a strict interior local maximum in the full four-instrument policy space.

Independent primitive-based differentiation gives a full gradient below $2.23 \times 10^{-16}$ in double precision. The 85-digit calculation, started from a perturbed nonuniform policy and wage vector, returns the same tariffs with a full gradient below $7.1 \times 10^{-85}$ and formula error below $4.0 \times 10^{-84}$. The exact arithmetic and curvature checks concern a strict local maximum; no claim that this example is a global maximum is needed for the qualification to Theorem 1.