Online Appendix

AI Displacement without Immiseration: A Ricardian Perspective on Jagged AI Growth

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A Technology Construction and the Date Equilibrium

This appendix derives the joint productivity law in equation (4) and the date-equilibrium results used in Section 2. We first construct persistent task-level productivities from Poisson processes over ideas. We then fix a date, derive task assignment and winning prices, characterize selection into worker production, and impose factor-market clearing. Any notation introduced solely to shorten the algebra is auxiliary and does not add an economic primitive.

We define the task processes on an atomless Fubini extension on which the exact law of large numbers applies. For the integrable task-level variables constructed below, cross-sectional averages equal their expectations almost surely. We use this result for assignment indicators and for the price moments in the final-good aggregator.

When $\sigma = 1$, we interpret the CES quantity and price indexes by their continuous Cobb--Douglas limits, $Y = \exp\{\int_0^1 \ln y(i)\, \mathrm{d}i\}$ and $P = \exp\{\int_0^1 \ln p(i)\, \mathrm{d}i\}$.

A.1 Poisson construction of ideas and nested-Fréchet productivity

We obtain the productivity law in three steps. First, a common task effect shifts the arrival intensity of both worker and AI ideas. Second, conditional on that effect, independent Poisson processes generate a fixed stock of worker ideas and a sequence of AI discoveries. Third, integrating out the task effect yields the joint nested-Fréchet law. Economically, task productivity equals the productivity of the best available idea: the worker frontier is fixed, whereas the AI frontier rises whenever a superior idea arrives. To obtain the joint distribution of these productivities, we adapt the marked-Poisson spectral representation used in Lind and Ramondo (2023*a*) and the symmetric logistic max-stable dependence studied in Lind and Ramondo (2023*b*). The economic mechanism differs from the knowledge-diffusion model in Lind and Ramondo (2023*a*): here, worker ideas remain fixed, AI ideas arrive directly for each task, and there is no diffusion between the two.

Let $v > 0$ denote the productivity of an idea. Let $\theta > 0$ denote the common marginal tail parameter and let $\rho \in [0, 1)$ govern the dependence between worker and AI productivity within a task. Define

$$\psi \equiv \frac{\theta}{1-\rho}, \qquad (1-\rho)\psi = \theta. \tag{A1}$$

The common marginal-tail parameter is θ. The Fréchet dispersion parameter ψ combines this tail behavior with within-task dependence and governs comparative advantage: a smaller ψ produces greater dispersion in z_L/z_M across tasks.

To introduce dependence within a task, for each task $i \in [0, 1]$ let the common task effect $S_i > 0$ scale both Poisson intensities. A larger S_i raises the intensity of worker and AI ideas at every productivity level. Conditional on S_i, the two idea processes are independent; unconditionally, their shared exposure to S_i makes worker and AI productivity dependent. The task effect S_i remains fixed over time. Across tasks, the variables S_i are independent and have Laplace transform

$$\mathbb{E}\left[e^{-uS_i}\right] = e^{-u^{1-\rho}}, \qquad u \geq 0. \tag{A2}$$

The parameter ρ therefore governs the dependence between worker and AI productivity. When $\rho = 0$, this transform implies $S_i = 1$ almost surely. For $0 < \rho < 1$, construct S_i as the sum of points $j > 0$ from a Poisson point process. Write $\nu_\rho(\,\mathrm{d}j) = [(1-\rho)/\Gamma(\rho)]j^{-2+\rho}\,\mathrm{d}j$ for its intensity measure. This measure satisfies

$$\frac{1-\rho}{\Gamma(\rho)} j^{-2+\rho}\,\mathrm{d}j, \qquad j > 0, \qquad \int_0^\infty \left(1 - e^{-uj}\right) \frac{1-\rho}{\Gamma(\rho)} j^{-2+\rho}\,\mathrm{d}j = u^{1-\rho}. \tag{A3}$$

The Poisson sum is well defined. Indeed,

$$\int_0^1 j\,\nu_\rho(\,\mathrm{d}j) = \frac{1-\rho}{\rho\Gamma(\rho)} < \infty, \qquad \nu_\rho([1,\infty)) = \frac{1}{\Gamma(\rho)} < \infty, \tag{A4}$$

The first bound implies that the total size of points below one is finite almost surely; the second implies that only finitely many points exceed one. Because $\nu_\rho((0, 1)) = \infty$, infinitely many positive small jumps occur almost surely. Consequently, $0 < S_i < \infty$ almost surely. Applying the Poisson exponential formula to equation (A3) gives equation (A2). For each point j in this process, draw independently a Poisson point process of worker ideas with intensity $jT_L\psi v^{-\psi-1}\,\mathrm{d}v$, a Poisson point process of AI ideas available at date zero with intensity $jT_{M,0}\psi v^{-\psi-1}\,\mathrm{d}v$, and a Poisson random measure of AI ideas arriving on dates $\tau > 0$ with intensity $j\,\mathrm{d}T_{M,\tau}\,\psi v^{-\psi-1}\,\mathrm{d}v$. Superposing over these points produces the conditional processes below and gives the marked-Poisson representation. We use the aggregate variable S_i to keep the derivation compact.

Conditional on S_i, worker ideas are drawn once from a Poisson point process over productivity levels $v > 0$ with intensity

$$S_i T_L \psi v^{-\psi-1}\,\mathrm{d}v, \tag{A5}$$

where $T_L > 0$ scales the intensity of worker ideas. If $\{v_{Lni}\}_n$ are their productivities, worker productivity is their maximum and remains fixed:

$$z_L(i) = \sup_n v_{Lni}. \tag{A6}$$

The conditional expected number of worker ideas with productivity above $x > 0$ is $S_i T_L x^{-\psi}$. Worker productivity is below x exactly when no worker idea has productivity above x. The Poisson probability of this event gives

$$\Pr[z_L(i) \le x \mid S_i] = \exp\left[-S_i T_L x^{-\psi}\right]. \tag{A7}$$

Thus the conditional worker distribution follows from the fixed process for worker ideas; it is not imposed separately.

New AI ideas for task i arrive at dates $\tau \ge 0$, and each idea has productivity $v > 0$. Conditional on S_i, the AI ideas available at date zero are drawn from a Poisson point process with intensity $S_i T_{M,0} \psi v^{-\psi-1}\, \mathrm{d}v$. After date zero, new AI ideas arrive according to a Poisson random measure over dates $\tau > 0$ and productivity levels $v > 0$, with intensity

$$S_i \ \mathrm{d}T_{M,\tau}\, \psi v^{-\psi-1}\, \mathrm{d}v. \tag{A8}$$

The state $T_{M,\tau} > 0$ records the cumulative intensity of AI ideas discovered by date τ; it is nondecreasing and right-continuous. The derivation treats the path of T_M as given; equation (20) later makes the growth of this path endogenous to research spending. If $\{(\tau_{ni}, v_{Mni})\}_n$ are the discovery dates and productivities of AI ideas for task i, AI productivity is the highest productivity discovered by date t:

$$z_{M,t}(i) = \sup_{n:\, \tau_{ni} \le t} v_{Mni}. \tag{A9}$$

Thus a new idea raises AI productivity only when it is better than the best idea already available, and the improvement then persists. At any finite date t, the intensity in (A8) accumulated through t has infinite mass near zero but finite mass above every positive productivity threshold. Hence, only finitely many ideas discovered by t can exceed any fixed positive incumbent. Conditional on S_i, the fixed process for worker ideas and the AI discovery process are independent. The resulting task processes are independent across i, and the exact law of large numbers equates task shares with the probabilities below.

By date t, the conditional expected number of AI ideas with productivity above $y > 0$ is $S_i T_{M,t} y^{-\psi}$. AI productivity is below y exactly when none of these ideas has productivity above y. Conditional independence and the Poisson probabilities that no worker idea exceeds x and no AI idea exceeds y give, for every $x, y > 0$,

$$\begin{aligned}\Pr\big[z_L(i) \le x,\ z_{M,t}(i) \le y \mid S_i\big] \\ = \exp\big[-S_i \left\{T_L x^{-\psi} + T_{M,t} y^{-\psi}\right\}\big]\,. \end{aligned} \tag{A10}$$

Integrating over S_i with (A2) therefore yields the nested-Fréchet cross section

$$\Pr\big[z_L(i) \le x,\ z_{M,t}(i) \le y\big] = \exp\Big[-\left\{T_L x^{-\psi} + T_{M,t} y^{-\psi}\right\}^{1-\rho}\Big]\,. \tag{A11}$$

This establishes equation (4) directly from the idea processes. The nested-Fréchet law is therefore an exact date-t implication of the marked-Poisson construction, not an additional distributional assumption or an asymptotic approximation. Setting either productivity threshold to infinity gives the marginal distributions,

$$\Pr[z_L(i) \le x] = \exp\Big[-T_L^{1-\rho} x^{-\theta}\Big]\,, \qquad \Pr[z_{M,t}(i) \le y] = \exp\Big[-T_{M,t}^{1-\rho} y^{-\theta}\Big]\,. \tag{A12}$$

Thus θ is the common marginal tail parameter, while $T_L^{1-\rho}$ and $T_{M,t}^{1-\rho}$ are the corresponding exponent scales.

Persistence is also explicit in the two-date distribution. For $0 \le t' < t$ and $x, y > 0$, every idea discovered by date t' must have productivity below $\min\{x, y\}$, whereas ideas arriving after t' need only have productivity below y. Conditional on S_i, Poisson counts on these two date intervals are independent. Applying the corresponding zero-count probabilities for the two date intervals and then integrating over S_i gives

$$\begin{aligned}\Pr\big[z_{M,t'}(i) \le x,\ z_{M,t}(i) \le y\big] \\ = \exp\Big[-\left\{T_{M,t'} \min\{x, y\}^{-\psi} + [T_{M,t} - T_{M,t'}] y^{-\psi}\right\}^{1-\rho}\Big]\,. \end{aligned} \tag{A13}$$

In particular, $z_{M,t}(i) \ge z_{M,t'}(i)$ almost surely. The dynamic model is therefore a persistent discovery process, not a sequence of independent cross-sectional draws.

A.2 Task assignment and winning prices

Fix a date and suppress time subscripts. At this date, factor supplies L and M, factor prices w and r, and technology scales T_L and T_M are all finite and strictly positive. Task prices satisfy (3). We maintain $0 < \sigma < 1 + \theta$; this is the necessary and sufficient restriction for the price index $P_t = \left[\int_0^1 p_t(i)^{1-\sigma}\, \mathrm{d}i\right]^{1/(1-\sigma)}$ to be finite.

The joint law gives assignment directly. Workers produce a task if and only if $z_L(i)/z_M(i) \ge w/r$. To derive the distribution of this ratio, condition on S_i and define $E_L = S_i T_L z_L^{-\psi}$ and $E_M = S_i T_M z_M^{-\psi}$.

The conditional Poisson laws imply that E_L and E_M are independent unit exponentials. Hence

$$\Pr\left[\frac{z_L}{z_M} \geq x \,\middle|\, S_i\right] = \Pr\left[E_M \geq \frac{T_M}{T_L} x^{\psi} E_L\right] = \int_0^{\infty} e^{-[1+(T_M/T_L)x^{\psi}]e} \, \mathrm{d}e.$$

The result does not depend on S_i, so integrating over the task effect gives the same expression. The common task effect changes both absolute productivities but leaves the distribution of comparative advantage unchanged:

$$\Pr\left[\frac{z_L(i)}{z_M(i)} \geq x\right] = \frac{T_L}{T_L + T_M x^{\psi}}, \qquad x > 0. \tag{A14}$$

The same formula also establishes the measured cost-advantage distribution used in equation (36). If $D(i) = [w/z_L(i)]/[r/z_M(i)]$, then $D(i) \leq d$ if and only if $z_L(i)/z_M(i) \geq w/(rd)$. Substituting $x = w/(rd)$ into equation (A14) gives

$$\Pr[D(i) \leq d] = \left[1 + \frac{T_M}{T_L}\left(\frac{w}{r}\right)^{\psi} d^{-\psi}\right]^{-1}.$$

Evaluating equation (A14) at $x = w/r$ gives

$$\pi_L = \Pr\left[\frac{z_L(i)}{z_M(i)} \geq \frac{w}{r}\right] = \frac{T_L}{T_L + T_M\left(\frac{w}{r}\right)^{\psi}}, \qquad \pi_M = 1 - \pi_L. \tag{A15}$$

Restoring time subscripts yields equation (5). The ratio $z_L(i)/z_M(i)$ has support $(0, \infty)$, so both factors win a positive task share at every finite positive relative price.

For the price calculations, define the following auxiliary cost-adjusted technique intensities:

$$A_L \equiv T_L w^{-\psi}, \qquad A_M \equiv T_M r^{-\psi}, \qquad A \equiv A_L + A_M. \tag{A16}$$

For any candidate task price $p > 0$, both factor-specific unit costs exceed p exactly when $z_L(i) < w/p$ and $z_M(i) < r/p$. Equation (A11) therefore gives

$$\begin{aligned}\Pr[p(i) > p] &= \Pr\left[z_L(i) < \frac{w}{p},\ z_M(i) < \frac{r}{p}\right] \\ &= \exp\left[-(Ap^{\psi})^{1-\rho}\right] = \exp\left[-A^{1-\rho}p^{\theta}\right].\end{aligned} \tag{A17}$$

The last equality uses $(1-\rho)\psi = \theta$. Thus the winning task price has density

$$\theta A^{1-\rho} p^{\theta-1} e^{-A^{1-\rho}p^{\theta}}, \qquad p > 0. \tag{A18}$$

We next derive the assignment shares and the conditional price distribution. Let $k \in \{L, M\}$ identify the factor that produces a task, and let A_k denote A_L or A_M accordingly. Conditional on S_i, define the transformed candidate costs $Q_L = (w/z_L)^{\psi}$ and $Q_M = (r/z_M)^{\psi}$. They are independent exponentials with respective rates $S_i A_L$ and $S_i A_M$. Therefore the conditional probability that factor k offers the lower unit cost and that the winning price does not exceed $\bar{p} > 0$ is

$$\int_0^{\bar{p}^{\psi}} S_i A_k e^{-S_i A q}\, \mathrm{d}q = \frac{A_k}{A}\left(1 - e^{-S_i A \bar{p}^{\psi}}\right).$$

Taking the expectation over S_i and using its Laplace transform, together with $(1-\rho)\psi = \theta$, gives

$$\Pr[k \text{ produces task } i,\ p(i) \le \bar{p}] = \frac{A_k}{A}\left(1 - e^{-A^{1-\rho}\bar{p}^{\theta}}\right), \qquad k \in \{L, M\}. \tag{A19}$$

Letting $\bar{p}$ tend to infinity recovers

$$\pi_L = \frac{A_L}{A} = \frac{T_L}{T_L + T_M\left(\frac{w}{r}\right)^{\psi}}, \qquad \pi_M = \frac{A_M}{A} = 1 - \pi_L. \tag{A20}$$

Dividing (A19) by the corresponding assignment share shows that the distribution of $p(i)$ is the same conditional on workers or AI producing the task. Thus the identity of the winner determines who performs the task but not the price distribution among the tasks that factor wins. Equation (A20) is exactly (6) with time subscripts suppressed.

The density in (A18) also gives the price index. For $\sigma \neq 1$, the exact law of large numbers implies

$$\begin{aligned} P^{1-\sigma} &= \int_0^{\infty} p^{1-\sigma}\theta A^{1-\rho} p^{\theta-1} e^{-A^{1-\rho}p^{\theta}}\, \mathrm{d}p \\ &= A^{-(1-\sigma)/\psi}\int_0^{\infty} u^{(1-\sigma)/\theta} e^{-u}\, \mathrm{d}u \\ &= A^{-(1-\sigma)/\psi}\Gamma\!\left(\frac{\theta+1-\sigma}{\theta}\right), \end{aligned} \tag{A21}$$

where the second line uses $u = A^{1-\rho}p^{\theta}$ and $(1-\rho)\psi = \theta$. Here $\Gamma(\cdot)$ is the gamma function. Define the fixed price-index constant

$$\gamma \equiv \begin{cases} \left[\Gamma\!\left(\dfrac{\theta+1-\sigma}{\theta}\right)\right]^{1/(1-\sigma)}, & \sigma \neq 1, \\ e^{-\gamma_{\mathrm{E}}/\theta}, & \sigma = 1, \end{cases} \tag{A22}$$

where γ_{E} is the Euler–Mascheroni constant. At $\sigma = 1$, the second line follows from $P = \exp\{\mathbb{E}[\ln p(i)]\}$ and is the continuous limit of the first line. For every $\sigma \in (0, 1+\theta)$, the price normalization is therefore

$$1 = \gamma A^{-1/\psi}. \tag{A23}$$

Equation (A23) separates the fixed moment adjustment γ, determined by θ and σ, from the cost-adjusted competitiveness term $A^{-1/\psi}$. The same parameter ψ governs how relative technology and factor prices shift task assignment.

A.3 Selection into worker production

Equation (A19) also determines the price distribution conditional on workers producing the task. Dividing by π_L and taking complements gives, for every $x > 0$,

$$\Pr[p(i) \geq x \mid \text{workers produce task } i] = \exp\left[-\left(T_L w^{-\psi} + T_M r^{-\psi}\right)^{1-\rho} x^{\theta}\right]. \tag{A24}$$

On a worker-produced task, competition implies $p(i) = w/z_L(i)$. Therefore

$$\begin{aligned}\Pr[z_L(i) \leq x \mid \text{workers produce task } i] &= \Pr[p(i) \geq w/x \mid \text{workers produce task } i] \\ &= \exp[-A^{1-\rho}(w/x)^{\theta}] \\ &= \exp[-T_L^{1-\rho}\pi_L^{-(1-\rho)}x^{-\theta}]. \end{aligned} \tag{A25}$$

The last equality uses $A = T_L w^{-\psi}/\pi_L$ from equation (A20) and $(1-\rho)\psi = \theta$.Selection raises worker productivity at every percentile by the same proportion, leaving relative dispersion across percentiles unchanged. The next calculation gives this proportional shift explicitly.

For $u \in (0,1)$, the unconditional marginal distribution in (A12) gives $Q_{z_L}(u) = T_L^{1/\psi}(-\ln u)^{-1/\theta}$. Solving (A25) at the same quantile and restoring time subscripts gives

$$\begin{aligned}Q_{z_L|L\,\text{wins}}(u) &= \pi_{L,t}^{-1/\psi} Q_{z_L}(u) \\ &= (-\ln u)^{-1/\theta} T_L^{1/\psi}\pi_{L,t}^{-1/\psi} \\ &= \gamma(-\ln u)^{-1/\theta} w_t. \end{aligned} \tag{A26}$$

The first equality shows that selection raises every fixed worker-productivity percentile by the factor $\pi_{L,t}^{-1/\psi}$ relative to the corresponding percentile across all tasks. The final equality follows from equations (A20) and (A23) and shows that every such percentile moves one-for-one with the real wage, exactly as stated in Theorem 1.a.

A.4 Factor clearing and aggregate output

This final step closes the date equilibrium. We first convert task shares into expenditure shares, then solve for the relative factor price, and finally aggregate output. Let $X \equiv Y$ denote aggregate expenditure; this alias is used only in the appendix derivations. CES demand for task i is $y(i) = Y[P/p(i)]^{\sigma}$, so task revenue is $YP^{\sigma}p(i)^{1-\sigma}$ and total expenditure is $Y = X$. Since the conditional winning-price distribution is the same for both factors, the average revenue of a winning task is also the same for both factors. The exact law of large numbers therefore makes each factor's expenditure share equal its task share. Constant returns and zero profits pay each task's revenue to the factor that produces it. Factor-market clearing is therefore

$$wL = \pi_L X, \qquad rM = \pi_M X, \qquad X = Y. \tag{A27}$$

Equation (A27) is the time-suppressed version of (7).

Dividing the first two equalities in (A27) and using (A20) gives

$$\left(\frac{w}{r}\right)^{1+\psi} = \frac{T_L}{T_M}\frac{M}{L}, \tag{A28}$$

which is (8). Equation (A28) has the usual scarcity interpretation: greater worker knowledge or AI capacity raises w/r, whereas greater AI knowledge or labor supply lowers it. The right-hand side is strictly positive and the left-hand side is strictly increasing in w/r, so the equilibrium relative price exists and is unique:

$$\frac{w}{r} = \left(\frac{T_L M}{T_M L}\right)^{1/(1+\psi)}.$$

With the final good as numeraire, this relative price, task pricing, CES demand, and factor clearing determine the date equilibrium. Without a numeraire, nominal prices are unique only up to a common scale. Substitution into the assignment odds yields

$$\frac{\pi_M}{\pi_L} = \frac{\widetilde{M}}{(T_L L^{\psi})^{1/(1+\psi)}}. \tag{A29}$$

Together with $\pi_L + \pi_M = 1$, this proves equation (9). The same objects give the real-wage formula in the main text directly. Since $A = A_L/\pi_L = T_L w^{-\psi}/\pi_L$, equation (A23) implies

$$\gamma w \left(\frac{\pi_L}{T_L}\right)^{1/\psi} = 1, \qquad w = \gamma^{-1}\left(\frac{T_L}{\pi_L}\right)^{1/\psi}. \tag{A30}$$

To shorten the remaining aggregation algebra, define the labor analogue of effective AI and the sum of the two effective factor sizes. Recalling the definition of $\widetilde{M}$ in Section 2, define

$$\widetilde{L} \equiv (T_L L^{\psi})^{1/(1+\psi)}, \qquad \widetilde{S} \equiv \widetilde{L} + \widetilde{M}. \tag{A31}$$

These symbols are derived composites, not additional primitives. Equations (9) and (A29) become

$$\pi_L = \frac{\widetilde{L}}{\widetilde{S}}, \qquad \pi_M = \frac{\widetilde{M}}{\widetilde{S}}. \tag{A32}$$

To derive aggregate output, use $w = \pi_L X/L$ from factor clearing in $A_L = T_L w^{-\psi}$. Substitution of (A31) and (A32) gives

$$A_L = \widetilde{L}\widetilde{S}^{\psi} X^{-\psi}, \qquad A_M = \widetilde{M}\widetilde{S}^{\psi} X^{-\psi}. \tag{A33}$$

Adding the two equalities yields $A = \widetilde{S}^{1+\psi} X^{-\psi}$. Substituting this expression into (A23) gives

$$Y = X = \gamma^{-1}\widetilde{S}^{(1+\psi)/\psi}, \tag{A34}$$

which is equation (10). Differentiating this aggregate production function with respect to L and M gives the equilibrium real factor prices:

$$\frac{\partial Y}{\partial L} = \gamma^{-1}\widetilde{S}^{1/\psi}\frac{\widetilde{L}}{L} = \frac{\pi_L Y}{L} = w, \qquad \frac{\partial Y}{\partial M} = \gamma^{-1}\widetilde{S}^{1/\psi}\frac{\widetilde{M}}{M} = \frac{\pi_M Y}{M} = r.$$

At any date at which T_M and M are positive and differentiable, log differentiation of effective AI, using the proportional-growth notation defined in Section 3.3, yields the primitive decomposition

$$g_{\widetilde{M},t} = \frac{1}{1+\psi}\left(g_{T_M,t} + \psi g_{M,t}\right). \tag{A35}$$

Equation (A35) shows that effective-AI growth is a weighted average of growth in AI knowledge and installed AI capacity, with weights $1/(1+\psi)$ and $\psi/(1+\psi)$. Both margins raise effective AI, but they receive different weights whenever $\psi \neq 1$.

A.5 Decomposition: Displacement vs Productivity Effects

This section derives the accounting form and the task demand form of Section 3. It shows that the elasticity of substitution between tasks does not determine the wage response, describes where aggregate productivity rises from, and shows how the result holds under the general distribution of Appendix B.2 and how it differs from the two models of Acemoglu and Restrepo (2018). The labor supply L and worker knowledge T_L are fixed while AI knowledge T_M or AI capacity M changes, and the final good is the numeraire.

Accounting form. Factor market clearing (A27) gives $wL = \pi_L Y$, so $w = \pi_L Y/L$. For any change, large or small, with L fixed,

$$\Delta \ln w = \Delta \ln(Y/L) + \Delta \ln \pi_L. \tag{A36}$$

For the first term, (A34) gives $Y = \gamma^{-1}\widetilde{S}^{(1+\psi)/\psi}$ and (A32) gives $\widetilde{S} = \widetilde{L}/\pi_L$ with $\widetilde{L} = (T_L L^{\psi})^{1/(1+\psi)}$ fixed, so Y/L is proportional to $(1/\pi_L)^{(1+\psi)/\psi}$ and

$$\Delta \ln(Y/L) = \frac{1+\psi}{\psi}\Delta \ln(1/\pi_L), \tag{A37}$$

which is (15). Substituting into (A36) gives $\Delta \ln w = (1/\psi)\Delta \ln(1/\pi_L)$, Theorem 1.a. Both expressions hold exactly for large changes.

Task demand form. Let $\mathcal{H}$ denote the set of tasks produced by workers, of measure π_L. On such a task, competition gives $p(i) = w/z_L(i)$ and demand gives $y(i) = Yp(i)^{-\sigma}$, so its revenue is $p(i)y(i) = Yw^{1-\sigma}z_L(i)^{\sigma-1}$, paid to labor. Adding over $\mathcal{H}$,

$$wL = Yw^{1-\sigma}\int_{\mathcal{H}} z_L(i)^{\sigma-1}\,\mathrm{d}i. \tag{A38}$$

The sum equals $\pi_L \bar{z}_L^{\sigma-1}$, where $\bar{z}_L^{\sigma-1}$ is the average of $z_L^{\sigma-1}$ across the tasks workers produce. $\bar{z}_L$ is the weighted power average of worker productivity on those tasks with exponent $\sigma - 1$. That is, $wL = Yw^{1-\sigma}\pi_L\bar{z}_L^{\sigma-1}$. The change with L fixed gives (14),

$$\sigma\,\Delta \ln w = \Delta \ln(Y/L) + \Delta \ln \pi_L + (\sigma-1)\,\Delta \ln \bar{z}_L. \tag{A39}$$

By (A25), worker productivity on the tasks workers produce is Fréchet with tail index θ and parameter $(T_L/\pi_L)^{1-\rho}$. For $\sigma < 1+\theta$ the average of $z_L^{\sigma-1}$ is finite, and with $(1-\rho)/\theta = 1/\psi$ from (A1),

$$\bar{z}_L^{\sigma-1} = \frac{1}{\pi_L}\int_{\mathcal{H}} z_L(i)^{\sigma-1}\,\mathrm{d}i = \Gamma\left(\frac{\theta+1-\sigma}{\theta}\right)\left(\frac{T_L}{\pi_L}\right)^{(\sigma-1)/\psi}. \tag{A40}$$

The change with T_L fixed gives

$$\Delta \ln \bar{z}_L = \frac{1}{\psi}\,\Delta \ln(1/\pi_L), \tag{A41}$$

for any value of σ: the distribution on the tasks workers keep is proportional to $(T_L/\pi_L)^{1/\psi}$, so every weighted power average changes with that factor. The change in labor's task demand is $\Delta \ln \pi_L + (\sigma-1)\Delta \ln \bar{z}_L = -(1 + \frac{1-\sigma}{\psi})\Delta \ln(1/\pi_L)$. The weight on displacement is positive because $\sigma < 1+\theta \le 1+\psi$. It is below one when tasks are substitutes. The tasks that move to AI are on average those on which workers are less productive, and with $\sigma > 1$ those tasks produce less revenue. The revenue weighted fall is therefore smaller than the fall in the task share. Substituting (A41) and (A37) into (A39) gives $\sigma\Delta \ln w = (\sigma/\psi)\Delta \ln(1/\pi_L)$, the result of Theorem 1.a.

The task elasticity does not determine the wage response. (A38) over Y gives labor's income share as $w^{1-\sigma}\pi_L\bar{z}_L^{\sigma-1} = \int_{\mathcal{H}} p(i)^{1-\sigma}\,\mathrm{d}i$. By (A19), the distribution of the price of a task among the tasks workers produce is the same as among the tasks AI produces, and so the same as among all tasks. With the final good as numeraire the average of $p^{1-\sigma}$ over all tasks is one, so the average over $\mathcal{H}$ is one as well, $\int_{\mathcal{H}} p(i)^{1-\sigma}\,\mathrm{d}i = \pi_L$, which is Corollary 1, and

$$\left(\frac{w}{\bar{z}_L}\right)^{1-\sigma} = 1, \qquad \text{that is,} \qquad w = \bar{z}_L. \tag{A42}$$

The price index of labor's task bundle equals the price of the final good, and the wage is the weighted power average of worker productivity on the tasks workers keep, with exponent $\sigma - 1$. At $\sigma = 1$ the same distribution makes the average of $\ln p$ over $\mathcal{H}$ zero, so $\ln w$ is the average of $\ln z_L$ over $\mathcal{H}$. (A42) then holds with $\bar{z}_L$ the limit of the power average as $\sigma \to 1$. (A40) shows the same directly: $\bar{z}_L = \Gamma(\frac{\theta+1-\sigma}{\theta})^{1/(\sigma-1)}(T_L/\pi_L)^{1/\psi} = \gamma^{-1}(T_L/\pi_L)^{1/\psi}$, which is the wage (11), because $\gamma^{1-\sigma} = \Gamma(\frac{\theta+1-\sigma}{\theta})$ by (A22). By (A42) the last term of (A39) is $(\sigma - 1)\Delta \ln w$, the negative of the demand response $(1 - \sigma)\Delta \ln w$ on the retained tasks. The productivity gain offsets the demand response, and (A39) reduces to (A36). σ is absent from the response of the wage to displacement for every value below $1 + \theta$. It works on the level of the wage only through γ.

Where aggregate productivity rises from. Let $\widetilde{M} = (T_M M^{\psi})^{1/(1+\psi)}$ as in (A31). The change in (A34) at fixed $\widetilde{L}$, using $\pi_M = \widetilde{M}/\widetilde{S}$, is

$$\mathrm{d}\ln Y = \frac{1+\psi}{\psi}\,\pi_M\,\mathrm{d}\ln\widetilde{M} = \frac{\pi_M}{\psi}\,\mathrm{d}\ln T_M + \pi_M\,\mathrm{d}\ln M. \tag{A43}$$

The first term is AI's expenditure share times $\mathrm{d}\ln T_M/\psi$, the proportional rise in AI productivity. This rise is the same in every task in the representation (4), where T_M works only through $z_M/T_M^{1/\psi}$. The second term is AI's income share times the growth of AI capacity. No term involves the tasks that change factor. At the marginal task the two factors' costs are the same, so the marginal change has no effect on output. In the aggregate, AI progress is a uniform rise in AI productivity across tasks. The gain in output is from the tasks AI performs before the change. Task by task the discovery process of Section 2 differs: a new AI idea can move a task from the worker with a strictly lower cost, (A9). The aggregate result is the same because output depends on the distribution of AI productivity only through T_M. Displacement is driven by the same growth, $\mathrm{d}\ln(1/\pi_L) = \pi_M\,\mathrm{d}\ln\widetilde{M}$ by (A32), so the ratio of the productivity effect to displacement in (A37) is fixed at $(1+\psi)/\psi$.

General distribution. In Appendix B.2, AI productivity is $z_M = A\tilde{z}_M$ with A its common level, comparative advantage is $a = z_L/\tilde{z}_M$, and the cutoff is $a^* = Aw/r$. The wage is $w = a^*/\Phi(a^*)$ with $\Phi(a^*)^{1-\sigma} = \mathbb{E}[\tilde{z}_M^{\sigma-1}\min\{a^*/a, 1\}^{1-\sigma}]$, and (B13) gives $\mathrm{d}\ln w/\mathrm{d}\ln a^* = s_M(a^*)$ with s_M AI's income share (B12). Φ is continuous in a^* at the cutoff because $\min\{a^*/a, 1\} = 1$ there. The task at the cutoff costs the same with either factor, so a change in the cutoff has no effect on output. Suppose instead that AI is confined to the tasks with $a \le a^c$ for some $a^c < a^*$, so that the marginal AI task is strictly cheaper with AI than with labor. Output at fixed factor supplies then responds to a change in that frontier through

$$\mathrm{d}\ln Y\big|_{L,M} = \frac{1}{1-\sigma}\left[\left(\frac{w}{a^c}\right)^{1-\sigma} - \left(\frac{r}{A}\right)^{1-\sigma}\right]\mathbb{E}[\tilde{z}_M^{\sigma-1} \mid a = a^c]\,g(a^c)\,\mathrm{d}a^c, \tag{A44}$$

which is positive for $a^c < a^*$ and zero at $a^c = a^*$. The tail elasticity η depends only on G, so σ works in the rate s_M/η only through the level of s_M. That is, σ works only through the gap between AI's income share and its task share π_M. Under the Fréchet distribution the rate s_M/η reduces to a constant. $s_M = \pi_M$ by Corollary 1, and (8) together with (A32) gives $\pi_L/\pi_M = (T_L/T_M)(w/r)^{-\psi}$, so π_L is log-logistic in the cutoff with $\eta = \psi\pi_M$ and $s_M/\eta = 1/\psi$.

The two models of Acemoglu and Restrepo (2018). Their model is within the general framework. The other factor is productive in every task at the same level, $\tilde{z}_M \equiv 1$ and $A = 1$. Labor's productivity is an increasing function $\gamma(i)$ over tasks $i \in [N-1, N]$, so $a = \gamma(i)$ and G is the distribution of γ under the uniform measure on tasks, with $g(a^c)\,\mathrm{d}a^c = \mathrm{d}I$ at $a^c = \gamma(I)$. Tasks with $i > I$ cannot be automated, and with task level inputs $\hat{\sigma}$ is used in the place of σ. To produce at the least cost the other factor would be on the tasks with $\gamma(i) \le W/R$, that is $i \le \tilde{I}$ with $\gamma(\tilde{I}) = W/R$, while technology lets $i \le I$, so it performs $i \le I^* = \min\{I, \tilde{I}\}$. The final good is the numeraire as here. With that numeraire $\int p(i)^{1-\hat{\sigma}}\,\mathrm{d}i = B^{1-\hat{\sigma}}$, and labor's income share is $s_L = B^{\hat{\sigma}-1}W^{1-\hat{\sigma}}\int_{I^*}^{N}\gamma(i)^{\hat{\sigma}-1}\,\mathrm{d}i$. With $n^* = N - I^*$ for labor's task share and $\bar{\gamma}$ for the weighted power average of γ over labor's tasks with exponent $\hat{\sigma}-1$, the corresponding expression to (A42) is

$$W = B\,\bar{\gamma}\left(\frac{n^*}{s_L}\right)^{\frac{1}{\hat{\sigma}-1}}. \tag{A45}$$

Their wage is labor productivity on retained tasks times a power of the ratio of labor's task share to its income share. The ratio is not one, because the other factor is productive in every task at the same level while labor is not. The task elasticity determines the wage response through it whenever automation changes it. When the frontier is the limit, $I < \tilde{I}$, the cutoff $a^c = \gamma(I)$ is below $a^* = W/R$. (A44) with $\tilde{z}_M \equiv 1$, $A = 1$, $g(a^c)\,\mathrm{d}a^c = \mathrm{d}I$ and their constant B is the productivity effect of automation in their Proposition 3,

$$\mathrm{d}\ln Y\big|_{K,L} = \frac{B^{\hat{\sigma}-1}}{1-\hat{\sigma}}\left[\left(\frac{W}{\gamma(I)}\right)^{1-\hat{\sigma}} - R^{1-\hat{\sigma}}\right]\mathrm{d}I, \qquad \mathrm{d}\ln W = \mathrm{d}\ln Y\big|_{K,L} - \frac{1-s_L}{\hat{\sigma}+\varepsilon_L}\Lambda_I\,\mathrm{d}I,$$

with ε_L the labor supply elasticity and $\Lambda_I = \gamma(I)^{\hat{\sigma}-1}/\int_I^N \gamma(i)^{\hat{\sigma}-1}\,\mathrm{d}i + 1/(I-N+1)$ the relative fall in task demand. The cost difference and $\hat{\sigma}$ both remain in the wage response. The wage falls when the cost difference is small. The ratio in (A45) also changes. With the labor supply fixed, $\varepsilon_L = 0$, $\mathrm{d}\ln s_L = \mathrm{d}\ln W - \mathrm{d}\ln Y|_{K,L}$ and $\mathrm{d}\ln n^* = -\mathrm{d}I/n^*$, so

$$\mathrm{d}\ln\frac{s_L}{n^*} = \left[\frac{1}{n^*} - \frac{(1-s_L)\Lambda_I}{\hat{\sigma}}\right]\mathrm{d}I,$$

which involves $\hat{\sigma}$ and is not zero in general. When comparative advantage is the limit, $\tilde{I} < I$, (A44) is zero, automation technology has no effect on factor prices, and the cutoff changes with the other factor,

$$\frac{\mathrm{d}\ln(W/R)}{\mathrm{d}\ln K} = \frac{1+\varepsilon_L}{\sigma_{\text{free}}+\varepsilon_L}, \qquad \sigma_{\text{free}} = \hat{\sigma} + \frac{\Lambda_I}{\varepsilon_\gamma}, \qquad \varepsilon_\gamma = \frac{\mathrm{d}\ln\gamma(I^*)}{\mathrm{d}I},$$

by their Proposition 2. The same argument works in their setting as it does here. With the final good as numeraire and $W = a^* R$, the change in their price index at the cutoff where costs are the same gives $\mathrm{d}\ln W/\mathrm{d}\ln a^* = s_K$, the income share of the other factor. Labor's task share is $n^* = N - \tilde{I}$. Under their Assumption 1′, $\gamma(i) = e^{\varepsilon_\gamma i}$ with ε_γ constant, the cutoff is $\tilde{I} = (\ln a^*)/\varepsilon_\gamma$, so $-\mathrm{d}\ln n^*/\mathrm{d}\ln a^* = 1/(\varepsilon_\gamma n^*)$ is the tail elasticity η. Thus

$$\frac{\mathrm{d}\ln W}{\mathrm{d}\ln(1/n^*)} = s_K\, \varepsilon_\gamma\, n^*, \tag{A46}$$

with $\hat{\sigma}$ absent from the formula. Given the fall in labor's task share, the task elasticity works in their model through the level of s_K, which differs from the task share $1 - n^*$ of the other factor exactly when the ratio in (A45) differs from one. It also works through the response of the cutoff to that factor, $1/\sigma_{\text{free}}$ at fixed labor supply. In this paper the corresponding expression is (8), which involves only ψ. Corollary 1 holds the ratio at one. The corresponding result to (A46) is Theorem 1.a, where $s_M = \pi_M$ and $\eta = \psi\pi_M$ make the rate constant.

B Proofs

The appendix restatement of each result records the complete hypotheses used in its proof; the main text suppresses some regularity conditions for brevity.

This appendix proves the theorems, corollaries, and mathematical boundaries introduced in the model sections of the main text. Throughout, $\widetilde{M}$ denotes effective AI, π_L and π_M denote task shares, and g_X denotes proportional growth, as defined in Sections 2 and 3.3. Each proof defines any additional notation locally.

B.1 Proofs of Theorem 1.a and Corollary 1

Remark (Authoritative appendix restatement of Theorem 1.a). Fix $T_L > 0$, $L > 0$, $\theta > 0$, and $\rho \in [0, 1)$, define the finite parameter $\psi = \theta/(1-\rho)$, and impose $0 < \sigma < 1 + \theta$. Maintain the joint distribution of labor and AI productivity constructed in Appendix A. Its assignment implication is equation (A15). At every date for which $T_{M,t}$ and M_t are finite and positive, $\pi_{L,t} > 0$ and the real wage is finite and satisfies

$$w_t = \gamma^{-1}\left(\frac{T_L}{\pi_{L,t}}\right)^{1/\psi}. \tag{B1}$$

Along any sequence of finite dates satisfying

$$\widetilde{M}_t \longrightarrow \infty, \tag{B2}$$

the labor task share and the real wage satisfy

$$\pi_{L,t} \longrightarrow 0, \qquad w_t \longrightarrow \infty. \tag{B3}$$

At every finite date and for each $u \in (0,1)$, the productivity quantile among tasks performed by workers satisfies

$$Q_{z_L|L\text{ wins}}(u) = \pi_{L,t}^{-1/\psi} Q_{z_L}(u) = \gamma(-\ln u)^{-1/\theta} w_t. \tag{B4}$$

Proof of Theorem 1.a. The proof has two steps. We first use the equilibrium task-share equation to establish finite-date interiority and the displacement limit. We then invoke the selection result derived from the full joint productivity law in Appendix A. The assignment equation alone would not determine quantiles of absolute worker productivity.

Finite dates and the displacement limit. We begin with the task share in equation (9). Written without any additional notation, the worker share is

$$\pi_{L,t} = \frac{[T_L L^{\psi}]^{1/(1+\psi)}}{[T_L L^{\psi}]^{1/(1+\psi)} + \widetilde{M}_t}. \tag{B5}$$

At any date at which $T_{M,t}$ and M_t are finite and positive, every term in (B5) is finite and positive. The denominator strictly exceeds the numerator, so $0 < \pi_{L,t} < 1$. Since $\gamma > 0$, $T_L > 0$, and $\psi > 0$, the real wage equation

$$w_t = \gamma^{-1} T_L^{1/\psi} \pi_{L,t}^{-1/\psi} \tag{B6}$$

then gives a finite positive real wage.

We next impose the path condition in equation (B2). The numerator of (B5) remains fixed and positive, whereas effective AI makes its denominator diverge. Hence $\pi_{L,t} \to 0$. Since $1/\psi > 0$, equation (B6) then implies $w_t \to \infty$. This proves both limits in equation (12).

Selected-worker productivity. Equation (A26) supplies the second step: selection rescales every worker-productivity quantile by $\pi_L^{-1/\psi}$, exactly as stated in the theorem. Combining this quantile identity with the finite-date and displacement results above proves Theorem 1.a. □

Remark (Authoritative appendix restatement of Corollary 1). Maintain the finite-date equilibrium constructed in Appendix A, including positive finite primitives and the identical conditional winning-price laws in equation (A19). Define labor's income share by

$$s_{L,t} \equiv \frac{w_t L}{Y_t} = \frac{w_t L}{w_t L + r_t M_t}. \tag{B7}$$

At every finite date, $s_{L,t} = \pi_{L,t}$.

Proof of Corollary 1. At every finite equilibrium, factor-market clearing in equation (7) gives

$$w_t L = \pi_{L,t} Y_t. \tag{B8}$$

All three factors on the right are positive, and Theorem 1.a gives $\pi_{L,t} > 0$. Dividing first by Y_t and then by $\pi_{L,t}$ gives

$$\frac{w_t L / Y_t}{\pi_{L,t}} = 1, \tag{B9}$$

which establishes $s_{L,t} = \pi_{L,t}$. □

B.2 General Distributions

Complete hypotheses for Theorem 1.b. Fix $\sigma > 0$ and a finite labor supply $L > 0$. At each date let the AI-service supply satisfy $0 < M_t < \infty$. Let $a = z_L/\tilde{z}_M > 0$ have a continuous distribution G with upper endpoint $\bar{a} \equiv \sup\{x : G(x) < 1\} \leq \infty$. On the interior of its support, assume that G has density $0 < g(a) < \infty$. For $\sigma \neq 1$, assume $0 < \mathbb{E}[\tilde{z}_M^{\sigma-1}] < \infty$ and $0 < \mathbb{E}[z_L^{\sigma-1}] < \infty$. For $\sigma = 1$, assume instead $\mathbb{E}|\ln \tilde{z}_M| + \mathbb{E}|\ln z_L| < \infty$. These conditions make the price index finite and justify the local differentiation and endpoint limits used below.

At every interior equilibrium cutoff, the elasticity in equation (16) is strictly positive. Define effective AI abundance relative to labor by $\mathcal{K}_t \equiv A_t M_t / L$. If $\mathcal{K}_t \to \infty$, then $a_t^* \to \bar{a}$ and $\pi_{L,t} \to 0$, while

$$w_t \longrightarrow \bar{a}\,\mathcal{Z}_M, \qquad \mathcal{Z}_M \equiv \begin{cases} \left(\mathbb{E}[\tilde{z}_M^{\sigma-1}]\right)^{1/(\sigma-1)}, & \sigma \neq 1, \\ \exp\{\mathbb{E}[\ln \tilde{z}_M]\}, & \sigma = 1. \end{cases} \tag{B10}$$

The limit is finite and positive when $\bar{a} < \infty$ and is infinite when $\bar{a} = \infty$. If $\mathcal{K}_t$ is nondecreasing, the equilibrium cutoff and real wage are nondecreasing in calendar time and labor's task share is nonincreasing. In particular, the main text's condition $A_t \to \infty$ implies $\mathcal{K}_t \to \infty$ under the standard path condition $\liminf_{t\to\infty} M_t > 0$. Without that restriction, the economically relevant condition is the product limit itself.

Throughout this subsection, $z_{M,t}(i) = A_t \tilde{z}_M(i)$, $a \equiv z_L/\tilde{z}_M$, and $a_t^* \equiv A_t w_t / r_t$. We allow arbitrary dependence between a and $\tilde{z}_M$, subject to the marginal and moment conditions stated above.

We proceed in four parts. First, we express the real wage in terms of the assignment cutoff. Second, we show that a higher cutoff raises the real wage, derive the wage–displacement elasticity, and establish the endpoint limit. Third, we characterize how the upper tail of G governs wage growth when relative advantage is unbounded. Finally, we derive the truncated path, and treat exact imitation as a separate boundary case.

The accounting representation. The competitive task price in equation (3) can be written as

$$p_t(i) = \min\left\{\frac{w_t}{z_L}, \frac{r_t}{A_t \tilde{z}_M}\right\} = \frac{w_t}{a_t^*}\tilde{z}_M^{-1} \min\left\{\frac{a_t^*}{a}, 1\right\}.$$

For $\sigma \neq 1$, define

$$\Phi(a^*) \equiv \mathbb{E}\left[\tilde{z}_M^{\sigma-1} \min\left\{\frac{a^*}{a}, 1\right\}^{1-\sigma}\right]^{1/(1-\sigma)}.$$

Substitution into the CES price index gives

$$\frac{w_t}{a_t^*}\Phi(a_t^*) = 1, \qquad w_t = \frac{a_t^*}{\Phi(a_t^*)}.$$

When $\sigma = 1$, the log price index gives

$$\Phi(a^*) = \exp\left\{-\mathbb{E}[\ln \tilde{z}_M] + \mathbb{E}\left[\ln \min\left\{\frac{a^*}{a}, 1\right\}\right]\right\},$$

and the same real-wage identity follows. Finally, continuity of G rules out a tie at the cutoff, so $\pi_{L,t} = 1 - G(a_t^*)$.

Monotonicity. The proof of part (i) rests on a pointwise scaling bound: for $0 < a^* \le a^{*\prime}$ and every $a > 0$, $\min\{a^{*\prime}/a, 1\} \le (a^{*\prime}/a^*)\min\{a^*/a, 1\}$; when $a^* < a^{*\prime}$, the inequality is strict if and only if $a < a^{*\prime}$. (If $a \ge a^{*\prime}$, both sides equal $a^{*\prime}/a$; if $a \le a^*$, the left side is 1 and the right side is $a^{*\prime}/a^* > 1$; if $a^* < a < a^{*\prime}$, the left side is 1 and the right side is $a^{*\prime}/a > 1$.) Raise the bound to the power $1 - \sigma$ and apply $\mathbb{E}[\tilde{z}_M^{\sigma-1}(\cdot)]^{1/(1-\sigma)}$. For $\sigma < 1$ both operations preserve the inequality; for $\sigma > 1$ each reverses it, so their composition preserves it; for $\sigma = 1$ take logs and expectations directly. In every case,

$$\Phi(a^{*\prime}) \le \frac{a^{*\prime}}{a^*}\,\Phi(a^*) \qquad \Longrightarrow \qquad \frac{a^{*\prime}}{\Phi(a^{*\prime})} \ge \frac{a^*}{\Phi(a^*)}. \tag{B11}$$

If in addition $G(a^{*\prime}) > 0$, the event $\{a < a^{*\prime}\}$ has positive probability, and since $\tilde{z}_M > 0$ almost surely, that event carries positive weight in the expectation; the strict case of the scaling bound then makes the first inequality in (B11) strict, and with it the second. Thus $a^*/\Phi(a^*)$ is strictly increasing wherever $G(a^*) > 0$: any advance of the cutoff, whatever drives it, raises the real wage. Note that the argument uses only monotonicity and linear homogeneity of the price aggregator, not the CES form: relative to the wage, the price of every retained task is unchanged and the price of every automated task strictly falls, so the workers' budget set expands.

The wage–displacement elasticity. Define AI's expenditure share at cutoff a^*,

$$s_M(a^*) \equiv \frac{\mathbb{E}\left[\tilde{z}_M^{\sigma-1}\mathbf{1}\{a \le a^*\}\right]}{\mathbb{E}\left[\tilde{z}_M^{\sigma-1}\min\{a^*/a, 1\}^{1-\sigma}\right]}. \tag{B12}$$

With constant returns and zero profits, expenditure shares are income shares, so $s_M = rM/Y$. For each a, the map $a^* \mapsto \min\{a^*/a, 1\}^{1-\sigma}$ is locally Lipschitz with almost-everywhere derivative $(1-\sigma)(a^*)^{-\sigma}a^{\sigma-1}\mathbf{1}\{a > a^*\}$. On any compact interval of positive cutoffs, the derivative is bounded by a fixed multiple of $\tilde{z}_M^{\sigma-1} + z_L^{\sigma-1}$. The standing moment assumptions therefore justify differentiating under the expectation. This gives

$$\frac{\mathrm{d}\ln[\Phi(a^*)^{1-\sigma}]}{\mathrm{d}\ln a^*} = (1-\sigma)[1-s_M(a^*)],$$
$$\frac{\mathrm{d}\ln w}{\mathrm{d}\ln a^*} = 1 - \frac{\mathrm{d}\ln\Phi}{\mathrm{d}\ln a^*} = s_M(a^*). \tag{B13}$$

This is an envelope result: relative to the wage, the prices of retained tasks do not move, while the prices of automated tasks fall one-for-one with the cutoff, so the elasticity of the real wage with respect to the cutoff is AI's expenditure share. (At $\sigma = 1$ the same computation gives $\mathrm{d}\ln\Phi/\mathrm{d}\ln a^* = 1 - G(a^*)$, that is, $s_M = \pi_M$: equal expenditure weights make AI's expenditure share its task share.) At an interior cutoff where G has density g with $0 < g(a^*) < \infty$, the task-share identity $\pi_L = 1 - G(a^*)$ gives $\mathrm{d}\ln(1/\pi_L)/\mathrm{d}\ln a^* = a^* g(a^*)/[1 - G(a^*)] \equiv \eta(a^*)$, and dividing (B13) by it yields (16). Since the cutoff is interior, $0 < \eta(a^*) < \infty$ and $s_M(a^*) > 0$, so this elasticity is strictly positive. This proves part (i).

The equilibrium cutoff and full displacement. The remaining step is to derive displacement from factor clearing rather than assume it. For an interior cutoff c, define

$$B_\sigma(c) \equiv \mathbb{E}\left[\tilde{z}_M^{\sigma-1}\mathbf{1}\{a \le c\}\right],$$
$$R_\sigma(c) \equiv \mathbb{E}\left[z_L^{\sigma-1}\mathbf{1}\{a > c\}\right] = \mathbb{E}\left[\tilde{z}_M^{\sigma-1}a^{\sigma-1}\mathbf{1}\{a > c\}\right]. \tag{B14}$$

For $\sigma = 1$, these expressions reduce to $B_1(c) = G(c)$ and $R_1(c) = 1 - G(c)$. The CES expenditure shares are

$$s_M(c) = \frac{B_\sigma(c)}{B_\sigma(c) + c^{1-\sigma}R_\sigma(c)}, \qquad s_L(c) = \frac{c^{1-\sigma}R_\sigma(c)}{B_\sigma(c) + c^{1-\sigma}R_\sigma(c)}. \tag{B15}$$

Factor clearing gives $wL = s_L Y$ and $rM = s_M Y$. Dividing these equations and using $c = Aw/r$ yields the scalar equilibrium condition

$$\mathcal{K} \equiv \frac{AM}{L} = F_\sigma(c) \equiv c\,\frac{s_M(c)}{s_L(c)} = c^\sigma \frac{B_\sigma(c)}{R_\sigma(c)}. \tag{B16}$$

The left side is effective AI abundance relative to labor. The function $F_\sigma(c)$ is the abundance ratio consistent with cutoff c.

The moment assumptions make B_σ and R_σ finite and continuous. On the support interior, B_σ is nondecreasing, R_σ is nonincreasing, and c^σ is strictly increasing. Thus F_σ is continuous and strictly increasing. At the lower endpoint it tends to zero; at $\bar{a}$ it tends to infinity because $R_\sigma(c) \to 0$. Equation (B16) therefore has a unique solution $a_t^* = F_\sigma^{-1}(\mathcal{K}_t)$. Consequently, $\mathcal{K}_t \to \infty$ implies $a_t^* \to \bar{a}$ and then $\pi_{L,t} = 1 - G(a_t^*) \to 0$. This is the full-displacement conclusion in part (ii).

The limit of the real wage. The preceding cutoff argument already establishes $a_t^* \to \bar{a}$ along any path with $\mathcal{K}_t \to \infty$. To evaluate the wage limit, since G is continuous, $a < \bar{a}$ almost surely, so

$\min\{a_t^*/a, 1\} \to 1$ task by task. For $\sigma < 1$, the price-moment integrand is bounded by $\tilde{z}_M^{\sigma-1}$. For $\sigma > 1$ and all sufficiently late cutoffs $a_t^* \geq c_0 > 0$, it is bounded by $\tilde{z}_M^{\sigma-1} + c_0^{1-\sigma} z_L^{\sigma-1}$. Dominated convergence therefore gives $\Phi(a_t^*) \to \mathbb{E}[\tilde{z}_M^{\sigma-1}]^{1/(1-\sigma)}$ for $\sigma \neq 1$. When $\sigma = 1$, $\ln \Phi(c) = -\mathbb{E}[\ln \tilde{z}_M] - \mathbb{E}[(\ln(a/c))_+]$ and the last expectation converges to zero by the logarithmic moment assumption. Hence $\Phi(a_t^*) \to \exp\{-\mathbb{E}[\ln \tilde{z}_M]\}$. Combining the two limits in $w_t = a_t^*/\Phi(a_t^*)$ proves (17), with the convention that the limit is $+\infty$ when $\bar{a} = \infty$. The limit is a finite positive plateau when $\bar{a} < \infty$ and diverges when $\bar{a} = \infty$. If $\mathcal{K}_t$ is nondecreasing, monotonicity in calendar time follows from equation (B16) and the wage monotonicity proved above. When $1 - G$ is regularly varying with index $-\psi$, the quantile $a_t^* = G^{-1}(1 - \pi_{L,t})$ is $\pi_{L,t}^{-1/\psi}$ times a slowly varying factor, and when the tail is lighter than every power, $a_t^* = o(\pi_{L,t}^{-\varepsilon})$ for every $\varepsilon > 0$; these shape results are the upper-tail trichotomy, stated and proved next.

Remark (The upper-tail trichotomy). Under the standing moment conditions, let G be strictly increasing in its upper tail. If its upper endpoint is infinite and $\pi_{L,t} \to 0$, then

$$w_t \sim \frac{a_t^*}{\Phi_\infty}, \qquad \Phi_\infty = \begin{cases} \left(\mathbb{E}[\tilde{z}_M^{\sigma-1}]\right)^{1/(1-\sigma)}, & \sigma \neq 1, \\ \exp\{-\mathbb{E}[\ln \tilde{z}_M]\}, & \sigma = 1. \end{cases} \tag{B17}$$

If $1 - G(a^*) = (a^*)^{-\psi} L(a^*)$ with L slowly varying, the wage is $\pi_{L,t}^{-1/\psi}$ times a slowly varying factor. Under the stronger tail equivalence $1 - G(a^*) \sim K(a^*)^{-\psi}$, it is asymptotic to $K^{1/\psi} \Phi_\infty^{-1} \pi_{L,t}^{-1/\psi}$. If the tail is lighter than every power, the wage diverges more slowly than every power of $1/\pi_{L,t}$. A finite endpoint instead gives the plateau in equation (17).

Proof of the upper-tail trichotomy. We verify the two unbounded-support branches in three steps: first the common price-index limit, then the regularly varying branch, and finally the lighter-than-every-power branch. The finite-endpoint branch is Theorem 1.b(ii).

The common price-index limit. Since G has infinite upper endpoint, every finite C satisfies $1 - G(C) > 0$. If a_t^* failed to diverge, a subsequence would satisfy $a_t^* \leq C$ for some finite C, and the identity $\pi_{L,t} = 1 - G(a_t^*)$ would imply $\pi_{L,t} \geq 1 - G(C) > 0$, a contradiction. Hence $a_t^* \to \infty$. For every task with finite $a(i)$, $\min\{a_t^*/a(i), 1\} \to 1$. The same dominated-convergence argument used in the endpoint proof gives $\Phi(a_t^*) \to \Phi_\infty$, proving equation (B17).

Regularly varying tails. Suppose next that $1 - G(a^*) = (a^*)^{-\psi} L(a^*)$, where L is slowly varying. Since G is strictly increasing in its upper tail, the cutoff is the upper quantile $G^{-1}(1 - \pi_{L,t})$. We verify the required inverse property directly. Define $U(x) \equiv G^{-1}(1 - 1/x)$ for sufficiently large x, fix $\lambda > 0$, and set $q \equiv \lambda^{1/\psi}$. Continuity and strict increase give

$$\frac{1 - G(U(\lambda x))}{1 - G(U(x))} = \frac{1}{\lambda} = q^{-\psi}. \tag{B18}$$

For any $\varepsilon \in (0, q)$, regular variation gives

$$\frac{1 - G((q-\varepsilon)U(x))}{1 - G(U(x))} \longrightarrow (q-\varepsilon)^{-\psi} > q^{-\psi},$$
$$\frac{1 - G((q+\varepsilon)U(x))}{1 - G(U(x))} \longrightarrow (q+\varepsilon)^{-\psi} < q^{-\psi}. \tag{B19}$$

Because the survival function is decreasing, these inequalities imply, for all sufficiently large x,

$$q - \varepsilon < \frac{U(\lambda x)}{U(x)} < q + \varepsilon. \tag{B20}$$

Letting $\varepsilon \downarrow 0$ proves $U(\lambda x)/U(x) \to \lambda^{1/\psi}$. Thus U is regularly varying with exponent $1/\psi$. Equivalently, for some slowly varying function ℓ,

$$a_t^* = \pi_{L,t}^{-1/\psi} \ell(1/\pi_{L,t}). \tag{B21}$$

Combining this equality with equation (B17) proves the first branch. Under $1 - G(a^*) \sim K(a^*)^{-\psi}$, the identity $\pi_{L,t} = 1 - G(a_t^*)$ gives $a_t^* \sim (K/\pi_{L,t})^{1/\psi}$, which yields the stated exact constant. The log-logistic relative-advantage law used in the paper satisfies this stronger tail equivalence.

Lighter-than-every-power tails. Finally, suppose that the tail is lighter than every power, meaning that $(a^*)^q[1 - G(a^*)] \to 0$ for every $q > 0$. Fix any $\varepsilon > 0$ and set $q = 1/\varepsilon$. The identity $\pi_{L,t} = 1 - G(a_t^*)$ gives

$$a_t^* \pi_{L,t}^{\varepsilon} = \left\{(a_t^*)^{1/\varepsilon}[1 - G(a_t^*)]\right\}^{\varepsilon} \longrightarrow 0. \tag{B22}$$

Thus $a_t^* = o(\pi_{L,t}^{-\varepsilon})$ for every $\varepsilon > 0$. Since $a_t^* \to \infty$ and $\Phi(a_t^*) \to \Phi_\infty \in (0, \infty)$, the real wage diverges, but more slowly than every power of $1/\pi_{L,t}$. This proves the remaining unbounded-support branch and hence the trichotomy. □

The truncated benchmark. We now cap relative advantage at $\bar{a}$. This truncated family provides a closed-form transition from the benchmark power law to the finite wage plateau in Theorem 1.b(ii). We impose independence between a and $\tilde{z}_M$ solely to obtain an explicit path. This assumption differs from the model's $\rho = 0$ case (independence in productivity levels makes $a = z_L/\tilde{z}_M$ and $\tilde{z}_M$ negatively dependent). When $\sigma = 1$, no independence restriction is needed because only the two marginal distributions enter Φ. To obtain the closed-form path, we truncate the benchmark's relative-advantage law, invert the task-share equation, evaluate the price index in the CES and log cases, and finally express the result in the units of Theorem 1.b.

At the reference date the benchmark's marginal law for relative advantage is log-logistic, by equation (A14). Define $\lambda_0 \equiv T_{M,0}/T_L$ and truncate at $\bar{a}$:

$$H(x) \equiv \frac{\lambda_0 x^{\psi}}{1 + \lambda_0 x^{\psi}}, \qquad G_{\bar{a}}(x) = \frac{H(x)}{H(\bar{a})}, \qquad 0 < x \leq \bar{a}. \tag{B23}$$

As $\bar{a} \to \infty$, $H(\bar{a}) \to 1$ and the family recovers the benchmark's marginal law at every x. The task-share equation $1 - \pi_L = H(a^*)/H(\bar{a})$ inverts to the explicit cutoff

$$a^*(\pi_L) = \bar{a}\left(\frac{1-\pi_L}{1+\lambda_0\bar{a}^{\psi}\pi_L}\right)^{1/\psi}, \qquad 0 < \pi_L < 1; \tag{B24}$$

substitution back into H confirms $H(a^*) = (1-\pi_L)H(\bar{a})$.

For $\sigma \neq 1$, independence factors the moment defining Φ:

$$\Phi(a^*)^{1-\sigma} = \mathbb{E}[\tilde{z}_M^{\sigma-1}]\,\mathbb{E}[\min\{a^*/a, 1\}^{1-\sigma}].$$

Write $\mathcal{B}_u^v(p,q) \equiv \int_u^v s^{p-1}(1-s)^{q-1}\,\mathrm{d}s$ for the interval incomplete beta integral. Splitting the remaining expectation at $a = a^*$ and substituting $s = H(a)$, under which $a^{\psi} = s/[\lambda_0(1-s)]$, gives

$$\Phi(a^*) = \left[\frac{\mathbb{E}[\tilde{z}_M^{\sigma-1}]}{H(\bar{a})}\left\{H(a^*) + (a^*)^{1-\sigma}\lambda_0^{(1-\sigma)/\psi}\,\mathcal{B}_{H(a^*)}^{H(\bar{a})}\left(1+\tfrac{\sigma-1}{\psi},\, 1+\tfrac{1-\sigma}{\psi}\right)\right\}\right]^{1/(1-\sigma)}. \tag{B25}$$

The interval integral is finite because its limits lie strictly inside the unit interval, and remains so when its first parameter is nonpositive. Equations (B24) and (B25) deliver the CES wage–displacement path through $w = a^*/\Phi(a^*)$. At $\sigma = 1$, the definition becomes

$$\ln\Phi(a^*) = -\mathbb{E}[\ln\tilde{z}_M] + \frac{1}{H(\bar{a})}\int_{a^*}^{\bar{a}} \ln(a^*/a)\;\mathrm{d}H(a),$$

for any dependence between a and $\tilde{z}_M$. Integration by parts then gives

$$\Phi(a^*) = \exp\left\{-\mathbb{E}[\ln\tilde{z}_M] + \ln\left(\frac{a^*}{\bar{a}}\right) + \frac{1}{\psi H(\bar{a})}\ln\left(\frac{1+\lambda_0\bar{a}^{\psi}}{1+\lambda_0(a^*)^{\psi}}\right)\right\}. \tag{B26}$$

The payoff comes from expressing the path in the units of Theorem 1.b. Differentiating (B23) gives the truncated family's tail elasticity, and with it the wage–displacement elasticity at $\sigma = 1$:

$$\eta(a^*) = \frac{\psi\,H(a^*)\,[1-H(a^*)]}{H(\bar{a}) - H(a^*)}, \qquad \frac{\mathrm{d}\ln w}{\mathrm{d}\ln(1/\pi_L)} = \frac{1}{\psi}\left[1 - \left(\frac{a^*}{\bar{a}}\right)^{\psi}\right] \quad (\sigma = 1), \tag{B27}$$

where the second expression follows from the first because $s_M = \pi_M = H(a^*)/H(\bar{a})$ at $\sigma = 1$. Equation (B27) shows where truncation bends the benchmark path. At every interior cutoff, truncation raises the tail elasticity above its benchmark value $\psi H(a^*)$. As $a^* \uparrow \bar{a}$, the tail elasticity diverges and the wage–displacement elasticity falls to zero, producing the finite plateau in Theorem 1.b(ii). Letting $\bar{a} \to \infty$ instead restores the benchmark elasticity $1/\psi$ at every task share. More generally, the correction factor is close to one whenever $a^*/\bar{a}$ is small. A bounded and an unbounded economy can therefore generate similar wage paths while the cutoff remains well below $\bar{a}$, even though their limiting wages differ. (For $\sigma \neq 1$ the path is (B25); the elasticity again vanishes at the endpoint, since $s_M \to 1$ while $\eta \to \infty$.)

Remark (Exact-imitation boundary). Under the separate technology $z_{M,t}(i) = A_t z_L(i)$ and the maintained finite price-moment condition, aggregate technology is linear in $L + A_t M_t$. Labor's real marginal product is a finite positive constant, and there is no assignment margin on which selection can operate.

Proof. The argument has two steps. We first collapse workers and AI into a common effective factor. We then aggregate across tasks and compute labor's real marginal product.

Suppose AI productivity is exactly proportional to worker productivity on every task:

$$z_{M,t}(i) = A_t z_L(i). \tag{B28}$$

The task technology in equation (2) then becomes

$$y_t(i) = z_L(i)\{\ell_t(i) + A_t m_t(i)\}. \tag{B29}$$

The total number of effective factor services is therefore

$$\int_0^1 \{\ell_t(i) + A_t m_t(i)\}\,\mathrm{d}i = L + A_t M_t. \tag{B30}$$

Conversely, any allocation of $L + A_t M_t$ effective services across tasks can be implemented by assigning the constant fractions $L/[L + A_t M_t]$ and $A_t M_t/[L + A_t M_t]$ of effective services to workers and AI, respectively. The economy is therefore equivalent to one with a single factor whose task productivity is $z_L(i)$.

Under the maintained finite price-moment condition, aggregation gives, for $\sigma \neq 1$,

$$Y_t = \left\{\int_0^1 z_L(i)^{\sigma-1}\,\mathrm{d}i\right\}^{1/(\sigma-1)} [L + A_t M_t]. \tag{B31}$$

At $\sigma = 1$, the first factor on the right of (B31) is replaced by $\exp\{\int_0^1 \ln z_L(i)\,\mathrm{d}i\}$. In either case, the first factor is a finite positive constant. Differentiating output with respect to L shows that labor's real marginal product equals that constant and is therefore finite. Relative productivity is the same on every task, so there is no assignment margin on which selection can operate.

For every fixed $\rho < 1$, by contrast, the productivity ratio in equation (A14) has support on $(0, \infty)$ and Theorem 1.a applies along its displacement path. Exact imitation at $\rho = 1$ is therefore a non-uniform boundary, not a failure of the selection result at a fixed $\rho < 1$. □

B.3 Proof of Equations (18) and (19)

Remark (Authoritative appendix restatement of Equations (18) and (19)). Maintain the finite-date equilibrium in Appendix A, hold $L > 0$ and $T_L > 0$ fixed, and let $\psi > 0$ be finite. At any date with positive finite $T_{M,t}$ and M_t, the level elasticity is

$$\frac{\partial \ln w_t}{\partial \ln \widetilde{M}_t} = \frac{\pi_{M,t}}{\psi}. \tag{B32}$$

Along any differentiable path through positive finite states,

$$\begin{aligned} g_{w,t} &= \frac{\pi_{M,t}}{\psi} g_{\widetilde{M},t}, \\ g_{Y,t} &= \frac{1+\psi}{\psi} \pi_{M,t} g_{\widetilde{M},t}, \\ g_{w,t} &= \frac{1}{1+\psi} g_{Y,t}. \end{aligned} \tag{B33}$$

The last identity remains valid when both growth rates are zero.

Proof of Equations (18) *and* (19). The proof has two steps. We first differentiate equilibrium levels with respect to effective AI. We then differentiate the same equations along an arbitrary differentiable path.

The level elasticity. Fix a date and suppress time subscripts. Holding L and T_L fixed, take logs of the worker task share in equation (9):

$$\begin{aligned} \ln \pi_L = &\frac{1}{1+\psi} \ln(T_L L^{\psi}) \\ &- \ln\left\{ (T_L L^{\psi})^{1/(1+\psi)} + \widetilde{M} \right\}. \end{aligned} \tag{B34}$$

Differentiate (B34) with respect to $\ln \widetilde{M}$. The first term is constant. The chain rule applied to the second term gives

$$\begin{aligned} \frac{\partial \ln \pi_L}{\partial \ln \widetilde{M}} &= -\frac{\widetilde{M}}{(T_L L^{\psi})^{1/(1+\psi)} + \widetilde{M}} \\ &= -\pi_M, \end{aligned} \tag{B35}$$

where the final equality uses equation (9).

The wage equation can be written as

$$\ln w = -\ln \gamma + \frac{1}{\psi}(\ln T_L - \ln \pi_L). \tag{B36}$$

Holding T_L fixed and substituting (B35) gives

$$\frac{\partial \ln w}{\partial \ln \widetilde{M}} = \frac{\pi_M}{\psi}, \tag{B37}$$

which is equation (18). Since $\psi > 0$, the derivative of this elasticity with respect to π_M is $1/\psi > 0$. The real wage response therefore rises with the fraction of tasks performed by AI.

Growth along a differentiable path. We now restore time subscripts. Direct log differentiation of effective AI gives

$$g_{\widetilde{M},t} = \frac{1}{1+\psi}\left[g_{T_M,t} + \psi g_{M,t}\right]. \tag{B38}$$

Since the worker term in equation (9) is fixed, the same chain-rule calculation as in (B35) yields

$$\begin{aligned} g_{\pi_L,t} &= -\pi_{M,t} g_{\widetilde{M},t} \\ &= -\frac{\pi_{M,t}}{1+\psi}\left[g_{T_M,t} + \psi g_{M,t}\right]. \end{aligned} \tag{B39}$$

Log differentiation of equation (11), followed by substitution of (B39), gives

$$\begin{aligned} g_{w,t} &= -\frac{1}{\psi} g_{\pi_L,t} \\ &= \frac{\pi_{M,t}}{\psi} g_{\widetilde{M},t} \\ &= \frac{\pi_{M,t}}{\psi(1+\psi)}\left[g_{T_M,t} + \psi g_{M,t}\right]. \end{aligned} \tag{B40}$$

This is equation (19).

Finally, log differentiation of aggregate output in equation (10) gives

$$\begin{aligned} g_{Y,t} &= \frac{1+\psi}{\psi} \pi_{M,t} g_{\widetilde{M},t} \\ &= \frac{\pi_{M,t}}{\psi}\left[g_{T_M,t} + \psi g_{M,t}\right], \end{aligned} \tag{B41}$$

where the second equality substitutes (B38). This proves the second line in equation (19).

The wage and output derivatives are multiples of the same bracketed primitive growth term. Comparing their coefficients, without dividing by a derivative that may be zero, gives

$$g_{w,t} = \frac{1}{1+\psi} g_{Y,t}. \tag{B42}$$

This completes the proof of Equations (18) and (19). □

B.4 Proof of Theorem 2

Remark (Authoritative appendix restatement of Theorem 2). Fix labor $L > 0$, the labor-productivity index $T_L > 0$, a finite Fréchet dispersion parameter $\psi > 0$, positive spending shares s_I and s_R with $s_I + s_R < 1$, depreciation $\delta \geq 0$, research productivity $\eta > 0$, and positive initial states M_0 and $T_{M,0}$. At each date, let output, the labor task share, and the real wage be determined by the competitive equilibrium in Section 2, and let the AI states follow equation (20). The state equations have a unique solution on $[0, t_\infty)$ for a finite maximal date t_∞, and no continuation beyond t_∞ has finite positive states. As $t \uparrow t_\infty$,

$$M_t \longrightarrow \infty, \qquad T_{M,t} \longrightarrow \infty, \qquad Y_t \longrightarrow \infty, \tag{B43}$$

and

$$\pi_{L,t} \asymp (t_\infty - t)^{\psi}, \qquad w_t \asymp (t_\infty - t)^{-1}. \tag{B44}$$

There is a date before t_∞ after which the real wage is strictly increasing; the result does not require the real wage to increase from the initial date. The real wage exponent is -1 for every admissible ψ.

Proof of Theorem 2. We organize the proof in six steps. Step 1 establishes the within-date equilibrium and a unique positive state path. Step 2 proves that the maximal date is finite. Step 3 shows that both AI states diverge at that date. Step 4 derives their limiting ratio. Step 5 obtains the asymptotic powers of the states, output, the labor task share, and the real wage. Step 6 proves that the real wage is eventually increasing.

Step 1: Within-date equilibrium, existence, and positivity. At each date, $T_{M,t}$ and M_t are predetermined. The feedback rule in (20) changes only these states and introduces no wedge into task prices. The within-date allocation is therefore the competitive equilibrium characterized in Section 2.

As long as both AI states are positive, output is

$$Y_t = \gamma^{-1} \left\{ [T_L L^{\psi}]^{1/(1+\psi)} + \widetilde{M}_t \right\}^{(1+\psi)/\psi}. \tag{B45}$$

This expression is continuously differentiable in $T_{M,t}$ and M_t when both states are positive. Hence the right-hand side of (20) is locally Lipschitz, and the standard existence and uniqueness theorem for ordinary differential equations gives a unique maximal solution. Research makes $T_{M,t}$ strictly increasing because $\dot{T}_{M,t} = \eta s_R Y_t > 0$. Moreover, $\dot{M}_t \geq -\delta M_t$, so comparison with exponential decay gives $M_t \geq M_0 e^{-\delta t} > 0$ before the maximal date. Both states therefore remain positive.

The output formula also gives two lower bounds that we use below:

$$Y_t \geq \gamma^{-1} T_L^{1/\psi} L > 0, \qquad Y_t \geq \gamma^{-1} T_{M,t}^{1/\psi} M_t. \tag{B46}$$

The first bound retains only the labor term in (B45); the second retains only the AI term. Both inequalities preserve direction because the outer exponent is positive.

Step 2: The maximal date is finite. Suppose, toward a contradiction, that the solution exists for every $t \geq 0$. We first use the output bounds to show that both AI states must become unbounded. We then derive a superlinear lower bound for installed-capacity growth, $\dot{M}_t \geq c_0 M_t^{1+1/\psi}$, which is incompatible with a finite state at every date. The first bound in (B46) gives

$$\dot{T}_{M,t} \geq \eta s_R \gamma^{-1} T_L^{1/\psi} L > 0, \tag{B47}$$

so $T_{M,t} \to \infty$. The second bound gives

$$g_{M,t} \geq s_I \gamma^{-1} T_{M,t}^{1/\psi} - \delta. \tag{B48}$$

The right-hand side tends to infinity. After a finite date it exceeds one, so $g_{M,t} \geq 1$ and M_t increases without bound.

On this increasing interval, division of the two state equations yields

$$\frac{\mathrm{d}T_M}{\mathrm{d}M} = \frac{\eta s_R Y}{s_I Y - \delta M} = \frac{\eta s_R / s_I}{1 - \delta M/(s_I Y)}. \tag{B49}$$

The denominator is positive because $\dot{M}_t > 0$. The output bound also gives

$$0 \leq \frac{\delta M_t}{s_I Y_t} \leq \frac{\delta \gamma}{s_I T_{M,t}^{1/\psi}} \longrightarrow 0. \tag{B50}$$

In particular, the denominator in (B49) is at most one, so $\mathrm{d}T_M/\mathrm{d}M \geq \eta s_R / s_I$. Integrating this inequality and using $M_t \to \infty$ gives, after another finite date,

$$T_{M,t} \geq \frac{\eta s_R}{2 s_I} M_t. \tag{B51}$$

After increasing the date once more, the preceding limit also gives $\delta M_t \leq s_I Y_t / 2$. Define the positive comparison constant

$$c_0 \triangleq \frac{s_I}{2\gamma} \left(\frac{\eta s_R}{2 s_I} \right)^{1/\psi} > 0. \tag{B52}$$

The capacity equation and the second output bound then imply

$$\begin{aligned} \dot{M}_t &= s_I Y_t - \delta M_t \\ &\geq \frac{s_I}{2\gamma} T_{M,t}^{1/\psi} M_t \\ &\geq c_0 M_t^{1+1/\psi}. \end{aligned} \tag{B53}$$

Multiplying the final inequality by the negative quantity $-(1/\psi)M_t^{-1-1/\psi}$ reverses its direction and gives

$$\frac{\mathrm{d}}{\mathrm{d}t}M_t^{-1/\psi} \leq -\frac{c_0}{\psi} < 0. \tag{B54}$$

Integration forces the positive variable $M_t^{-1/\psi}$ to reach zero in finite time. This contradicts global existence. The maximal date t_∞ is therefore finite.

Step 3: Both AI states diverge. At least one state must become unbounded as t approaches t_∞. Otherwise, positivity from Step 1 and the continuation theorem would extend the solution beyond its maximal date. The state equations satisfy the exact identity

$$\dot{M}_t + \delta M_t = \frac{s_I}{\eta s_R}\dot{T}_{M,t}. \tag{B55}$$

Integrating this identity gives

$$\frac{s_I}{\eta s_R}T_{M,t} = M_t - M_0 + \frac{s_I}{\eta s_R}T_{M,0} + \delta \int_0^t M_u \,\mathrm{d}u. \tag{B56}$$

If M_t is unbounded, the right-hand side makes $T_{M,t}$ unbounded as well. Conversely, if $T_{M,t}$ were unbounded while M_t remained bounded, the right-hand side would remain bounded on the finite interval $[0, t_\infty)$, contradicting the equality. Thus both states are unbounded. The state $T_{M,t}$ is increasing, so it converges to infinity. Once $T_{M,t}$ is sufficiently large, (B48) makes M_t increasing. Its unboundedness then implies $M_t \to \infty$ as $t \uparrow t_\infty$.

Step 4: The state ratio. We next show that cumulative depreciation is negligible relative to $T_{M,t}$. Rearranging (B56) gives

$$M_t - \frac{s_I}{\eta s_R}T_{M,t} = M_0 - \frac{s_I}{\eta s_R}T_{M,0} - \delta \int_0^t M_u \,\mathrm{d}u. \tag{B57}$$

Define

$$K \triangleq \max\left\{M_0 - \frac{s_I}{\eta s_R}T_{M,0}, 0\right\} \geq 0. \tag{B58}$$

Since the integral in (B57) is nonnegative,

$$M_t \leq \frac{s_I}{\eta s_R}T_{M,t} + K. \tag{B59}$$

Fix $\varepsilon \in (0, t_\infty)$ and split the depreciation integral at $t_\infty - \varepsilon$. The integral over $[0, t_\infty - \varepsilon]$ is finite and becomes negligible after division by $T_{M,t}$ because $T_{M,t} \to \infty$. On the remaining interval,

$T_{M,u} \leq T_{M,t}$ by monotonicity. Equation (B59) therefore gives

$$\frac{1}{T_{M,t}} \int_{t_\infty - \varepsilon}^{t} M_u \, \mathrm{d}u \leq \varepsilon \left[\frac{s_I}{\eta s_R} + \frac{K}{T_{M,t}} \right]. \tag{B60}$$

First let $t \uparrow t_\infty$ and then let $\varepsilon \downarrow 0$. The two parts of the integral satisfy

$$\frac{\int_0^t M_u \, \mathrm{d}u}{T_{M,t}} \longrightarrow 0. \tag{B61}$$

Dividing (B57) by $T_{M,t}$ now yields the local state-ratio result

$$\frac{T_{M,t}}{M_t} \longrightarrow \frac{\eta s_R}{s_I}. \tag{B62}$$

Step 5: Asymptotic powers. We first show that effective AI dominates the fixed labor term in aggregate output. Equation (B62) gives

$$\frac{\widetilde{M}_t}{M_t} = \left[\frac{T_{M,t}}{M_t} \right]^{1/(1+\psi)} \longrightarrow \left(\frac{\eta s_R}{s_I} \right)^{1/(1+\psi)} > 0. \tag{B63}$$

Because $M_t \to \infty$, effective AI diverges while $[T_L L^\psi]^{1/(1+\psi)}$ remains fixed. Substitution into (B45) yields

$$\frac{Y_t}{\gamma^{-1} T_{M,t}^{1/\psi} M_t} = \left\{ 1 + \frac{[T_L L^\psi]^{1/(1+\psi)}}{\widetilde{M}_t} \right\}^{(1+\psi)/\psi} \longrightarrow 1. \tag{B64}$$

This limit and $T_{M,t} \to \infty$ imply $Y_t / M_t \to \infty$, and hence $\delta M_t / Y_t \to 0$.

Define the positive limiting coefficient

$$c_\infty \triangleq s_I \gamma^{-1} \left(\frac{\eta s_R}{s_I} \right)^{1/\psi} > 0. \tag{B65}$$

Using the capacity equation, the machine-dominance limit, and the state-ratio limit gives

$$\begin{aligned} \frac{\dot{M}_t}{M_t^{1+1/\psi}} &= s_I \frac{Y_t}{M_t^{1+1/\psi}} - \delta M_t^{-1/\psi} \\ &\longrightarrow c_\infty. \end{aligned} \tag{B66}$$

Direct differentiation therefore gives

$$\frac{\mathrm{d}}{\mathrm{d}t} M_t^{-1/\psi} = -\frac{1}{\psi} M_t^{-1-1/\psi} \dot{M}_t \longrightarrow -\frac{c_\infty}{\psi}. \tag{B67}$$

Since $M_t^{-1/\psi} \to 0$ as $t \uparrow t_\infty$, integration backward from the terminal date yields

$$\frac{M_t^{-1/\psi}}{t_\infty - t} \longrightarrow \frac{c_\infty}{\psi}. \tag{B68}$$

Consequently,

$$M_t \sim \left(\frac{\psi}{c_\infty}\right)^{\psi} (t_\infty - t)^{-\psi}, \qquad T_{M,t} \sim \frac{\eta s_R}{s_I} \left(\frac{\psi}{c_\infty}\right)^{\psi} (t_\infty - t)^{-\psi}. \tag{B69}$$

Combining these state powers with (B64) also gives $Y_t \asymp (t_\infty - t)^{-(1+\psi)}$.

It remains to translate the state powers into the labor task share and the real wage. Equation (9) gives

$$\begin{aligned} \pi_{L,t} &= \frac{[T_L L^\psi]^{1/(1+\psi)}}{[T_L L^\psi]^{1/(1+\psi)} + \widetilde{M}_t} \\ &\sim \frac{[T_L L^\psi]^{1/(1+\psi)}}{(\eta s_R / s_I)^{1/(1+\psi)} M_t} \asymp (t_\infty - t)^{\psi}. \end{aligned} \tag{B70}$$

Substitution into (11) then gives

$$w_t \asymp \pi_{L,t}^{-1/\psi} \asymp (t_\infty - t)^{-1}. \tag{B71}$$

These are the two asymptotic claims in (21).

Step 6: Eventual wage growth. Because $T_{M,t} \to \infty$, (B48) implies $g_{M,t} > 0$ after a finite date. The research equation gives $g_{T_M,t} > 0$ at every date. Hence, after that finite date,

$$g_{T_M,t} + \psi g_{M,t} > 0. \tag{B72}$$

Equation (19) then implies

$$g_{w,t} > 0. \tag{B73}$$

Before that date, depreciation can make $g_{M,t}$ sufficiently negative to reverse the sign. The closure therefore guarantees eventual, but not all-date, wage growth. Combining Steps 1–6 proves Theorem 2. □

B.5 Real-wage growth under jagged AI adoption

This subsection proves Theorem 3 and Corollary 2 from Section 4. The theorem characterizes the limiting ratio of real-wage growth to output growth, labor's long-run income share, and the sectors in which employment concentrates when AI productivity grows at different rates across sectors. The corollary compares balanced adoption across all sectors with jagged adoption that leaves some sectors unchanged.

Factor payments determine the sectoral allocation of labor and AI services:

$$\frac{L_{s,t}}{L} = \frac{\beta_s[1-\pi_{Ms,t}]}{\sum_{k=1}^{S}\beta_k[1-\pi_{Mk,t}]}, \qquad \frac{M_{s,t}}{M_t} = \frac{\beta_s \pi_{Ms,t}}{\sum_{k=1}^{S}\beta_k \pi_{Mk,t}}. \tag{B74}$$

Equation (B74) shows that employment and the use of AI services adjust across sectors as task assignment changes; neither is fixed in advance within a sector.

Remark (Theorem 3: Real-wage growth under jagged AI adoption). Consider the finite-sector Cobb–Douglas economy in Section 4. Factor supplies L and M are fixed and positive; sectoral expenditure weights β_s are positive and sum to one; the worker- and AI-productivity parameters T_{Ls} and $T_{Ms,0}$ are positive; and $\psi > 0$ is finite. Let $\nu_s \geq 0$ for every sector, with at least one $\nu_s > 0$, and set $T_{Ms,t} = T_{Ms,0}A_t^{\nu_s}$. Let the common AI-productivity index $A_t > 0$ be differentiable. It satisfies $A_0 = 1$ and $\mathrm{d}A_t/\mathrm{d}t \geq 0$ at every date and tends to infinity. In addition, $\mathrm{d}A_t/\mathrm{d}t > 0$ at all sufficiently late dates. Then

$$\frac{g_w}{g_Y} \longrightarrow 1 - \frac{\psi}{1+\psi}\frac{\min_s \nu_s}{\sum_s \beta_s \nu_s} \in \left[\frac{1}{1+\psi}, 1\right]. \tag{B75}$$

If every ν_s is positive, labor's income share converges to zero and employment concentrates in the sectors with the slowest AI-productivity growth. If at least one ν_s is zero, labor's income share converges to a positive limit, and the employment share of every sector in which AI productivity grows converges to zero.

Comparison normalization. A comparison across adoption profiles requires a common scale. We therefore hold fixed $\bar{\nu} \equiv \sum_s \beta_s \nu_s > 0$ and the path A_t. The limiting real-wage elasticity and its ratio to output growth are

$$\frac{\mathrm{d}\ln w}{\mathrm{d}\ln A} \longrightarrow \frac{\bar{\nu}}{\psi} - \frac{\min_s \nu_s}{1+\psi}, \qquad \frac{g_w}{g_Y} \longrightarrow 1 - \frac{\psi}{1+\psi}\frac{\min_s \nu_s}{\bar{\nu}}. \tag{B76}$$

Because $\min_s \nu_s \leq \bar{\nu}$ and every $\beta_s > 0$, both expressions are minimized uniquely by balanced adoption, $\nu_s = \bar{\nu}$ for every sector. This is the precise sense in which balanced adoption is worst for long-run real-wage growth. Without holding $\bar{\nu}$ fixed, the theorem ranks only wage growth relative to output growth.

Remark (Even-progress case of Corollary 2). Under the assumptions of Theorem 3, suppose AI productivity grows at the same rate in every sector. Thus, $T_{Ms,t} = T_{Ms,0}A_t^{\nu}$ for the same $\nu > 0$ in every sector. Then

$$\frac{g_w}{g_Y} \longrightarrow \frac{1}{1+\psi}, \tag{B77}$$

and labor's income share converges to zero. Because every sector has the same AI-productivity growth rate, the limiting ratio of real-wage growth to output growth reaches the theorem's lower bound, $1/(1+\psi)$.

Remark (Jagged-adoption case of Corollary 2). Under the assumptions of Theorem 3, suppose AI productivity grows in some sectors and remains fixed in others. Thus, $T_{Ms,t} = T_{Ms,0}A_t^{\nu_s}$, with $\nu_s > 0$ in some sectors and $\nu_s = 0$ in the remaining sectors. Then

$$\frac{g_w}{g_Y} \longrightarrow 1. \tag{B78}$$

Labor's income share converges to a positive limit, and the employment share of every sector in which AI productivity grows converges to zero.

Proof of Theorem 3 and Corollary 2. Because A_t is increasing at all sufficiently late dates, we first differentiate the equilibrium with respect to $\ln A$; the chain rule then converts these derivatives into calendar-time growth rates. The proof proceeds in four steps. First, we derive how AI task shares, the relative factor price, real wages, and output respond to the common AI-productivity index. Second, we study the case in which AI productivity grows in every sector. Third, we study the case in which AI productivity grows in some sectors and remains fixed in others. Finally, we establish the bounds in equation (27).

Step 1. Elasticity identities. From equations (23) and (26), sector s's AI task share satisfies

$$\frac{\mathrm{d}\pi_{Ms}}{\mathrm{d}\ln A} = \pi_{Ms}(1-\pi_{Ms})\left[\nu_s + \psi\frac{\mathrm{d}\ln(w/r)}{\mathrm{d}\ln A}\right]. \tag{B79}$$

Differentiating the factor-market-clearing equation (24) with respect to $\ln A$ and solving for the response of w/r gives

$$\frac{\mathrm{d}\ln(w/r)}{\mathrm{d}\ln A} = -\frac{\sum_s \beta_s\pi_{Ms}(1-\pi_{Ms})\nu_s}{\left(\sum_s \beta_s\pi_{Ms}\right)\left(1-\sum_s \beta_s\pi_{Ms}\right)+\psi\sum_s \beta_s\pi_{Ms}(1-\pi_{Ms})}. \tag{B80}$$

The right side of the factor-market-clearing equation is $M/[M+(w/r)L]$. Its derivative with respect to $\ln(w/r)$ is

$$-\frac{M}{M+(w/r)L}\left[1-\frac{M}{M+(w/r)L}\right]. \tag{B81}$$

At every finite date, the aggregate AI task share lies strictly between zero and one, while the remaining terms in the denominator of (B80) are nonnegative. The denominator is therefore positive, so the equilibrium relative factor price is differentiable with respect to $\ln A$ at every finite date.

Differentiating the real-wage expression in equation (25) gives

$$\frac{\mathrm{d}\ln w}{\mathrm{d}\ln A} = \frac{1}{\psi}\sum_s \beta_s\pi_{Ms}\nu_s + \left(\sum_s \beta_s\pi_{Ms}\right)\frac{\mathrm{d}\ln(w/r)}{\mathrm{d}\ln A}. \tag{B82}$$

The derivative of $\ln[L + M/(w/r)]$ with respect to $\ln(w/r)$ is $-M/[M + (w/r)L]$. Substituting factor-market clearing into the derivative of output cancels the relative-factor-price term in (B82) and leaves

$$\frac{\mathrm{d}\ln Y}{\mathrm{d}\ln A} = \frac{1}{\psi}\sum_s \beta_s \pi_{Ms} \nu_s. \tag{B83}$$

Step 2. AI productivity grows in every sector. Suppose first that ν_s is positive in every sector. Define the expenditure-weighted labor-to-AI productivity ratio by

$$I(A) \equiv \sum_s \beta_s \frac{T_{Ls}}{T_{Ms,0}A^{\nu_s}} = \sum_s \beta_s \frac{T_{Ls}}{T_{Ms,t}}. \tag{B84}$$

Because there are finitely many sectors and $T_{Ms,t} = T_{Ms,0}A_t^{\nu_s}$, the terms with the smallest exponent decline most slowly and determine the limit:

$$I(A) \sim \left[\sum_{s:\, \nu_s = \min_k \nu_k} \beta_s \frac{T_{Ls}}{T_{Ms,0}}\right] A^{-\min_s \nu_s}. \tag{B85}$$

In particular, $I(A) \to 0$.

The following function will approximate the equilibrium relative factor price:

$$b(A) \equiv \left[\frac{M}{L} I(A)\right]^{1/(1+\psi)}. \tag{B86}$$

Written in terms of sectoral productivity and the factor-price ratio, labor's share of total factor income satisfies

$$\frac{(w/r)L}{M + (w/r)L} = \sum_s \beta_s \frac{T_{Ls}/T_{Ms,t}}{T_{Ls}/T_{Ms,t} + (w/r)^{\psi}}. \tag{B87}$$

Equations (B85) and (B86) imply

$$\max_s \frac{T_{Ls}/T_{Ms,t}}{b(A)^{\psi}} \longrightarrow 0. \tag{B88}$$

For every sector, $T_{Ls}/T_{Ms,t}$ is no greater than $CA^{-\min_s \nu_s}$ for some finite constant C that does not depend on the sector or on A. By contrast, for large A, $b(A)^{\psi}$ declines in proportion to $A^{-\psi \min_s \nu_s/(1+\psi)}$. Because $\psi/(1+\psi) < 1$, the ratio tends to zero.

Fix $0 < \underline{\lambda} < \overline{\lambda} < \infty$ and evaluate equation (B87) at $w/r = \lambda b(A)$, where $\lambda \in [\underline{\lambda}, \overline{\lambda}]$. Uniformly over this interval, the labor-income side equals $\lambda b(A)L/M[1 + o(1)]$, while the technology side equals $I(A)[\lambda b(A)]^{-\psi}[1 + o(1)]$. The technology side divided by the labor-income side is therefore

$$\lambda^{-(1+\psi)}[1 + o(1)], \tag{B89}$$

where we used $b(A)^{1+\psi} = (M/L)I(A)$. The left side of (B87) increases in w/r, while the right side decreases in w/r. For any fixed $\varepsilon \in (0, 1)$, choose an interval of this form containing both $1 - \varepsilon$ and

$1+\varepsilon$. The right side eventually exceeds the left side at $(1-\varepsilon)b(A)$ and falls below the left side at $(1+\varepsilon)b(A)$ for all sufficiently large A. The unique solution therefore lies between these values. Letting $\varepsilon \downarrow 0$ proves

$$\frac{w}{r} \sim \left[\frac{M}{L}\sum_s \beta_s \frac{T_{Ls}}{T_{Ms,t}}\right]^{1/(1+\psi)} \asymp A^{-\min_s \nu_s/(1+\psi)}. \tag{B90}$$

Equation (B90) implies $\pi_{Ms} \to 1$ in every sector and

$$\frac{wL}{wL+rM} = \frac{(w/r)L}{M+(w/r)L} \longrightarrow 0. \tag{B91}$$

The same relative-factor-price limit determines where workers are employed. From equation (B74),

$$\frac{L_s}{L} \sim \frac{\beta_s T_{Ls}/T_{Ms,t}}{\sum_k \beta_k T_{Lk}/T_{Mk,t}}. \tag{B92}$$

For every sector with $\nu_s > \min_k \nu_k$, the right side converges to zero. Employment therefore concentrates in the sectors with the slowest AI-productivity growth.

We next evaluate the response of w/r in equation (B80).Equation (B92) implies that the normalized weights $\beta_s\pi_{Ms}(1-\pi_{Ms})$ concentrate on the same slowest-growing sectors. Since $\pi_{Ms} \to 1$ in every sector, the corresponding weighted average of the exponents converges to their common minimum:

$$\frac{\sum_s \beta_s\pi_{Ms}(1-\pi_{Ms})\nu_s}{\sum_s \beta_s\pi_{Ms}(1-\pi_{Ms})} \longrightarrow \min_s \nu_s. \tag{B93}$$

Moreover,

$$0 \le \frac{\sum_s \beta_s(1-\pi_{Ms})^2}{\sum_s \beta_s(1-\pi_{Ms})} \le \max_s(1-\pi_{Ms}) \longrightarrow 0. \tag{B94}$$

Since

$$\sum_s \beta_s\pi_{Ms}(1-\pi_{Ms}) = \sum_s \beta_s(1-\pi_{Ms}) - \sum_s \beta_s(1-\pi_{Ms})^2, \tag{B95}$$

the preceding bound yields

$$\frac{\left(\sum_s \beta_s\pi_{Ms}\right)\left(1-\sum_s \beta_s\pi_{Ms}\right)}{\sum_s \beta_s\pi_{Ms}(1-\pi_{Ms})} \longrightarrow 1. \tag{B96}$$

Substituting equations (B93) and (B96) into (B80) gives

$$\frac{\mathrm{d}\ln(w/r)}{\mathrm{d}\ln A} \longrightarrow -\frac{\min_s \nu_s}{1+\psi}. \tag{B97}$$

Finally, $\pi_{Ms} \to 1$ and $\sum_s \beta_s = 1$, so equations (B82) and (B83) imply

$$\frac{\mathrm{d}\ln Y}{\mathrm{d}\ln A} \longrightarrow \frac{1}{\psi}\sum_s \beta_s \nu_s, \tag{B98}$$

$$\frac{\mathrm{d}\ln w}{\mathrm{d}\ln A} \longrightarrow \frac{1}{\psi}\sum_s \beta_s \nu_s - \frac{\min_s \nu_s}{1+\psi}. \tag{B99}$$

Dividing the limiting real-wage response by the limiting output response proves equation (27) when AI productivity grows in every sector.

Step 3. AI productivity grows in some sectors and remains fixed in others. Suppose next that AI productivity grows in some but not all sectors. Let $\mathcal{K}$ denote the sectors in which AI productivity grows:

$$\mathcal{K} \equiv \{s : \nu_s > 0\}. \tag{B100}$$

Let $\mathcal{N}$ denote the sectors in which AI productivity remains fixed, so that $\mathcal{N} \equiv \{s : \nu_s = 0\}$. Both sets are nonempty. For any fixed $0 < q_- < q_+ < \infty$, the largest difference over $w/r \in [q_-, q_+]$ between the left side of equation (24) and the expression below converges to zero:

$$\sum_{s\in\mathcal{K}} \beta_s + \sum_{s\in\mathcal{N}} \beta_s \frac{T_{Ms,0}(w/r)^{\psi}}{T_{Ls} + T_{Ms,0}(w/r)^{\psi}}. \tag{B101}$$

This limiting condition has a direct economic interpretation: advancing sectors spend asymptotically their entire factor payment on AI, whereas fixed-productivity sectors retain an endogenous division between labor and AI.Define the limiting relative factor price $(w/r)_\infty$ as the unique positive solution to

$$\sum_{s\in\mathcal{K}} \beta_s + \sum_{s\in\mathcal{N}} \beta_s \frac{T_{Ms,0}(w/r)_\infty^{\psi}}{T_{Ls} + T_{Ms,0}(w/r)_\infty^{\psi}} = \frac{M}{M + (w/r)_\infty L}. \tag{B102}$$

This solution exists and is unique. The left side is strictly increasing and the right side is strictly decreasing. As $w/r \downarrow 0$, the left side converges to $\sum_{s\in\mathcal{K}} \beta_s < 1$ while the right side converges to one. As $w/r \uparrow \infty$, the left side converges to one while the right side converges to zero.

Below $(w/r)_\infty$, the left side of (B102) is smaller than the right side; above $(w/r)_\infty$, the left side is larger. The preceding bound on the approximation error implies that these inequalities also hold in the finite-A factor-market-clearing equation when A is sufficiently large. Because the left side increases and the right side decreases with w/r, the equilibrium lies between any such lower and upper values. Letting these values approach $(w/r)_\infty$ proves

$$\frac{w}{r} \longrightarrow (w/r)_\infty \in (0, \infty). \tag{B103}$$

Consequently,

$$\frac{wL}{wL + rM} \longrightarrow \frac{(w/r)_\infty L}{M + (w/r)_\infty L} > 0. \tag{B104}$$

For each $s \in \mathcal{K}$, $1 - \pi_{Ms} \to 0$. By factor clearing, the denominator in the labor-allocation equation (B74) converges to $(w/r)_\infty L/[M + (w/r)_\infty L] > 0$, the limiting share of income paid to labor. Therefore, $L_s/L \to 0$ in every sector in which AI productivity grows.

In the numerator of equation (B80), the terms with $s \in \mathcal{N}$ are zero because $\nu_s = 0$, while the terms with $s \in \mathcal{K}$ converge to zero because $\pi_{Ms} \to 1$. The denominator of equation (B80) has a positive lower bound for all sufficiently large A because

$$\left(\sum_s \beta_s \pi_{Ms}\right)\left(1 - \sum_s \beta_s \pi_{Ms}\right) \longrightarrow \frac{M}{M + (w/r)_\infty L}\left[1 - \frac{M}{M + (w/r)_\infty L}\right] > 0. \tag{B105}$$

It follows that

$$\frac{\mathrm{d}\ln(w/r)}{\mathrm{d}\ln A} \longrightarrow 0. \tag{B106}$$

AI productivity grows only in sectors $s \in \mathcal{K}$, and AI's task share in each of these sectors converges to one. Equations (B82) and (B83) therefore yield

$$\frac{\mathrm{d}\ln Y}{\mathrm{d}\ln A} \longrightarrow \frac{1}{\psi}\sum_s \beta_s \nu_s, \tag{B107}$$

$$\frac{\mathrm{d}\ln w}{\mathrm{d}\ln A} \longrightarrow \frac{1}{\psi}\sum_s \beta_s \nu_s. \tag{B108}$$

The ratio of the real-wage response to the output response converges to one, as stated in equation (27) when $\min_s \nu_s = 0$.

Step 4. Growth-rate bounds and the two endpoint cases. Because the expenditure weights are positive and sum to one, the expenditure-weighted average exponent is at least as large as the smallest exponent:

$$0 \le \min_s \nu_s \le \sum_s \beta_s \nu_s. \tag{B109}$$

The weighted average is strictly positive because AI productivity grows in at least one sector. Equation (27) therefore places the limiting ratio in the interval $[1/(1+\psi), 1]$. When AI productivity grows in every sector, the lower endpoint is reached only if $\min_s \nu_s = \sum_s \beta_s \nu_s$. Because every β_s is positive, this equality holds exactly when all sectors have the same positive exponent. The upper endpoint is reached when AI productivity grows in at least one sector and remains fixed in at least one other sector, because then $\min_s \nu_s = 0$. These are, respectively, the balanced- and jagged-adoption cases of Corollary 2.

At all sufficiently late dates, $g_A > 0$. The calendar-time growth rates of real wages and output equal their respective derivatives with respect to $\ln A$ multiplied by g_A. This common factor therefore cancels from their ratio:

$$\frac{g_w}{g_Y} = \frac{\mathrm{d}\ln w/\mathrm{d}\ln A}{\mathrm{d}\ln Y/\mathrm{d}\ln A}. \tag{B110}$$

Together, the preceding limits prove Theorem 3 and the balanced- and jagged-adoption cases of Corollary 2. □

B.6 Upper-tier CES boundary

Replace equation (22) with a constant-elasticity-of-substitution (CES) aggregator. Let $\sigma_U \in (0,1)$ denote the elasticity of substitution across sectoral goods and set

$$Y = \left[\sum_s \beta_s^{1/\sigma_U} Y_s^{(\sigma_U-1)/\sigma_U}\right]^{\sigma_U/(\sigma_U-1)}, \qquad 1 = \left[\sum_s \beta_s P_s^{1-\sigma_U}\right]^{1/(1-\sigma_U)}. \tag{B111}$$

Sector s's expenditure share, which is now endogenous, is

$$\frac{P_s Y_s}{Y} = \frac{\beta_s P_s^{1-\sigma_U}}{\sum_{k=1}^{S} \beta_k P_k^{1-\sigma_U}}. \tag{B112}$$

Accordingly, factor-market clearing becomes

$$\sum_s \frac{\beta_s P_s^{1-\sigma_U}}{\sum_{k=1}^{S} \beta_k P_k^{1-\sigma_U}} \pi_{Ms} = \frac{M}{M + (w/r)L}. \tag{B113}$$

Proof of the CES boundary. Suppose the final-good aggregator is the CES aggregator in equation (B111), with $\sigma_U \in (0,1)$. Suppose also that $\nu_s > 0$ in a nonempty strict subset of sectors and $\nu_s = 0$ in the remaining sectors. The proof has four steps. We first establish existence and uniqueness at every finite date. We then construct a compact interval that contains the relative factor price at all sufficiently late dates. This construction lets us derive the limiting economy formed by the sectors with fixed AI intensity without first assuming that the relative factor price converges. The final step evaluates the limiting real wage and output.

CES expenditure shares and finite-date equilibrium. For any positive trial value of w/r and any $A \geq 1$, define sector s's endogenous expenditure share by

$$\omega_s(w/r, A) \equiv \frac{\beta_s \gamma_s^{1-\sigma_U} [T_{Ls} + T_{Ms,0} A^{\nu_s} (w/r)^{\psi}]^{-(1-\sigma_U)/\psi}}{\sum_j \beta_j \gamma_j^{1-\sigma_U} [T_{Lj} + T_{Mj,0} A^{\nu_j} (w/r)^{\psi}]^{-(1-\sigma_U)/\psi}}. \tag{B114}$$

This expression follows by substituting the sector price in equation (23) into the CES expenditure-share equation (B112); the common wage cancels.

Define the aggregate share of expenditure paid to AI services at the trial relative factor price by

$$\mathcal{M}(w/r, A) \equiv \sum_s \omega_s(w/r, A) \frac{T_{Ms,0} A^{\nu_s} (w/r)^\psi}{T_{Ls} + T_{Ms,0} A^{\nu_s} (w/r)^\psi}. \tag{B115}$$

Define the corresponding factor-balance map by

$$\mathcal{R}(w/r, A) \equiv \frac{w}{r} \frac{\mathcal{M}(w/r, A)}{1 - \mathcal{M}(w/r, A)}. \tag{B116}$$

The CES clearing equation (B113) is equivalent to

$$\mathcal{R}(w/r, A) = \frac{M}{L}. \tag{B117}$$

We next show that the factor-balance map is strictly increasing. In the next two definitions, π_{Ms} is evaluated at the same trial value of w/r and the same A as $\omega_s(w/r, A)$. Define the map's within-sector term by

$$\mathcal{W}(w/r, A) \equiv \sum_s \omega_s(w/r, A) \pi_{Ms}(1 - \pi_{Ms}) > 0, \tag{B118}$$

and define its between-sector term by

$$\mathcal{V}(w/r, A) \equiv \sum_s \omega_s(w/r, A) [\pi_{Ms} - \mathcal{M}(w/r, A)]^2 \geq 0. \tag{B119}$$

Direct differentiation gives

$$\frac{\mathrm{d} \ln(P_s/w)}{\mathrm{d} \ln(w/r)} = -\pi_{Ms}, \tag{B120}$$

$$\frac{\mathrm{d} \ln \omega_s}{\mathrm{d} \ln(w/r)} = (1 - \sigma_U)[\mathcal{M}(w/r, A) - \pi_{Ms}], \tag{B121}$$

$$\frac{\mathrm{d} \pi_{Ms}}{\mathrm{d} \ln(w/r)} = \psi \pi_{Ms}(1 - \pi_{Ms}). \tag{B122}$$

It follows that

$$\frac{\mathrm{d} \mathcal{M}}{\mathrm{d} \ln(w/r)} = \psi \mathcal{W} - (1 - \sigma_U) \mathcal{V}. \tag{B123}$$

The identity

$$\mathcal{M}(1 - \mathcal{M}) = \mathcal{W} + \mathcal{V} \tag{B124}$$

follows by expanding the square in the definition of $\mathcal{V}$. Combining equations (B123) and (B124) yields

$$\frac{\mathrm{d} \ln \mathcal{R}}{\mathrm{d} \ln(w/r)} = \frac{(1 + \psi) \mathcal{W} + \sigma_U \mathcal{V}}{\mathcal{W} + \mathcal{V}} > 0. \tag{B125}$$

At any finite A, all human and AI intensities are positive. As $w/r \downarrow 0$, the largest sectoral AI share is of order $(w/r)^{\psi}$. Thus $\mathcal{M}(w/r, A)$ is at most of that order and $\mathcal{R}(w/r, A) \to 0$. As $w/r \uparrow \infty$, the largest sectoral labor task share is of order $(w/r)^{-\psi}$. Hence

$$1 - \mathcal{M}(w/r, A) \leq \max_{s}(1 - \pi_{Ms}) = O((w/r)^{-\psi}), \tag{B126}$$

which implies $\mathcal{R}(w/r, A) \to \infty$. Continuity and strict monotonicity therefore give a unique positive solution to (B117) at every finite date.

The fixed-intensity economy and an explicit compact interval. Define the nonempty set of advancing sectors by $\mathcal{K} \equiv \{s : \nu_s > 0\}$ and the nonempty set of sectors with fixed AI intensity by $\mathcal{N} \equiv \{s : \nu_s = 0\}$. For a positive trial value of w/r, define the fixed-intensity economy's expenditure share for each $n \in \mathcal{N}$ by

$$\omega_n^{\mathcal{N}}(w/r) \equiv \frac{\beta_n \gamma_n^{1-\sigma_U} [T_{Ln} + T_{Mn,0}(w/r)^{\psi}]^{-(1-\sigma_U)/\psi}}{\sum_{j \in \mathcal{N}} \beta_j \gamma_j^{1-\sigma_U} [T_{Lj} + T_{Mj,0}(w/r)^{\psi}]^{-(1-\sigma_U)/\psi}}. \tag{B127}$$

Define the fixed-intensity economy's aggregate AI expenditure share by

$$\mathcal{M}_{\mathcal{N}}(w/r) \equiv \sum_{n \in \mathcal{N}} \omega_n^{\mathcal{N}}(w/r) \frac{T_{Mn,0}(w/r)^{\psi}}{T_{Ln} + T_{Mn,0}(w/r)^{\psi}}, \tag{B128}$$

and define the corresponding factor-balance map by

$$\mathcal{R}_{\mathcal{N}}(w/r) \equiv \frac{w}{r} \frac{\mathcal{M}_{\mathcal{N}}(w/r)}{1 - \mathcal{M}_{\mathcal{N}}(w/r)}. \tag{B129}$$

The same finite-economy argument applies here. Hence $\mathcal{R}_{\mathcal{N}}$ is continuous and strictly increasing from zero to infinity. Define $(w/r)_{\mathcal{N}}$ as the unique positive solution to

$$\mathcal{R}_{\mathcal{N}}((w/r)_{\mathcal{N}}) = \frac{M}{L}. \tag{B130}$$

We now construct the compact interval before considering any sequence of full equilibria. Define its lower and upper endpoints by

$$c_{-} \equiv \frac{1}{2}(w/r)_{\mathcal{N}}, \qquad c_{+} \equiv 2(w/r)_{\mathcal{N}}. \tag{B131}$$

Strict monotonicity makes the two factor-balance gaps

$$\delta_{-} \equiv \frac{M}{L} - \mathcal{R}_{\mathcal{N}}(c_{-}) > 0, \qquad \delta_{+} \equiv \mathcal{R}_{\mathcal{N}}(c_{+}) - \frac{M}{L} > 0. \tag{B132}$$

For every $w/r \in [c_-, c_+]$, the full-economy expenditure share of the advancing sectors obeys the explicit bound

$$\sum_{k\in\mathcal{K}} \omega_k(w/r, A) \leq \frac{\sum_{k\in\mathcal{K}} \beta_k \gamma_k^{1-\sigma_U} [T_{Mk,0} A^{\nu_k} c_-^{\psi}]^{-(1-\sigma_U)/\psi}}{\sum_{n\in\mathcal{N}} \beta_n \gamma_n^{1-\sigma_U} [T_{Ln} + T_{Mn,0} c_+^{\psi}]^{-(1-\sigma_U)/\psi}} \longrightarrow 0. \tag{B133}$$

To obtain the numerator bound, use $w/r \geq c_-$ and drop the positive human intensity from each advancing sector's bracket. To obtain the denominator bound, retain only the residual sectors, use $w/r \leq c_+$, and recall that the exponent $-(1-\sigma_U)/\psi$ is negative. The denominator displayed in (B133) is a fixed positive number. Every term in its numerator converges to zero because $\nu_k > 0$ for $k \in \mathcal{K}$. Thus the convergence is uniform on $[c_-, c_+]$.

For a residual sector $n \in \mathcal{N}$, its full-economy expenditure share can be written as

$$\omega_n(w/r, A) = \left[1 - \sum_{k\in\mathcal{K}} \omega_k(w/r, A)\right] \omega_n^{\mathcal{N}}(w/r). \tag{B134}$$

Because every task share lies in $[0, 1]$, this identity gives the explicit comparison

$$|\mathcal{M}(w/r, A) - \mathcal{M}_{\mathcal{N}}(w/r)| \leq 2 \sum_{k\in\mathcal{K}} \omega_k(w/r, A). \tag{B135}$$

Define the smallest labor expenditure share in the fixed-intensity economy on the compact interval by

$$d \equiv \min_{w/r\in[c_-,c_+]} [1 - \mathcal{M}_{\mathcal{N}}(w/r)] > 0. \tag{B136}$$

The strict inequality follows from continuity and the positive human intensity in every fixed-intensity sector. For all sufficiently large A, equation (B135) is smaller than $d/2$ uniformly on the same interval. Consequently,

$$|\mathcal{R}(w/r, A) - \mathcal{R}_{\mathcal{N}}(w/r)| = \frac{w}{r} \frac{|\mathcal{M}(w/r, A) - \mathcal{M}_{\mathcal{N}}(w/r)|}{[1 - \mathcal{M}(w/r, A)][1 - \mathcal{M}_{\mathcal{N}}(w/r)]} \tag{B137}$$

$$\leq \frac{2c_+}{d^2} |\mathcal{M}(w/r, A) - \mathcal{M}_{\mathcal{N}}(w/r)| \longrightarrow 0 \tag{B138}$$

uniformly for $w/r \in [c_-, c_+]$.

Choose A large enough that the absolute difference in (B138) is smaller than $\min\{\delta_-, \delta_+\}/2$ throughout the interval. At its two endpoints,

$$\mathcal{R}(c_-, A) < \mathcal{R}_{\mathcal{N}}(c_-) + \frac{\delta_-}{2} = \frac{M}{L} - \frac{\delta_-}{2} < \frac{M}{L}, \tag{B139}$$

$$\mathcal{R}(c_+, A) > \mathcal{R}_{\mathcal{N}}(c_+) - \frac{\delta_+}{2} = \frac{M}{L} + \frac{\delta_+}{2} > \frac{M}{L}. \tag{B140}$$

Strict monotonicity of the full-economy factor-balance map now puts its unique equilibrium solution inside $[c_-, c_+]$. This establishes compactness without first assuming convergence of the relative factor price.

Convergence to the fixed-intensity economy. The equilibrium relative factor price converges to $(w/r)_{\mathcal{N}}$. To see this directly, choose any two positive values inside $[c_-, c_+]$, one strictly below $(w/r)_{\mathcal{N}}$ and the other strictly above it. Strict monotonicity of $\mathcal{R}_{\mathcal{N}}$ gives a positive gap from M/L at each value. Uniform convergence in (B138) makes the full-economy map have the same signs relative to M/L for all sufficiently large A. The equilibrium is therefore bracketed between the two values. Taking these values arbitrarily close to $(w/r)_{\mathcal{N}}$ proves

$$\frac{w}{r} \longrightarrow (w/r)_{\mathcal{N}}. \tag{B141}$$

Together with equation (B133), this limit also proves that

$$\sum_{k \in \mathcal{K}} \omega_k(w/r, A) \longrightarrow 0 \tag{B142}$$

at the equilibrium relative factor price.

Limiting wages and output. The price normalization and the sector prices in equation (23) give the real wage directly:

$$w = \left\{ \sum_s \beta_s \gamma_s^{1-\sigma_U} [T_{Ls} + T_{Ms,0} A^{\nu_s} (w/r)^{\psi}]^{-(1-\sigma_U)/\psi} \right\}^{-1/(1-\sigma_U)}. \tag{B143}$$

The terms for $s \in \mathcal{K}$ converge to zero. The terms for $n \in \mathcal{N}$ converge by (B141). Therefore

$$w \longrightarrow \left\{ \sum_{n \in \mathcal{N}} \beta_n \gamma_n^{1-\sigma_U} [T_{Ln} + T_{Mn,0} (w/r)_{\mathcal{N}}^{\psi}]^{-(1-\sigma_U)/\psi} \right\}^{-1/(1-\sigma_U)} \in (0, \infty). \tag{B144}$$

Finally, the income identity in equation (25) implies

$$Y \longrightarrow \left(\lim_{A \to \infty} w \right) \left[L + \frac{M}{(w/r)_{\mathcal{N}}} \right] \in (0, \infty). \tag{B145}$$

Thus an upper-tier elasticity below one makes both the real wage and output converge to finite positive limits when only a strict subset of sectors advances. This result establishes the boundary stated in Section 4; it does not determine a limit for the ratio of their growth rates. □

B.7 Worker exposure and sectoral sorting

This subsection develops the worker-incidence result in three stages. First, we derive sufficient-statistics formulas for group real income and aggregate output, both in levels and in exact changes. Second, we establish equilibrium existence and derive bounds on sectoral labor supply and aggregate income. Third, we combine these bounds to establish the mobility floor and prove Theorem 4. Throughout, wages are measured per efficiency unit of labor, and ratios to the AI rental are written explicitly when they enter the proof. To match the notation in Section 5, an evaluation bar with subscript $\kappa \downarrow 1$ is a mnemonic label for the specific-factors benchmark, not a limit of the Fréchet worker economy. The benchmark is a separate economy in which sector s has a fixed efficiency-unit labor endowment $\ell_s^0 > 0$ chosen to reproduce baseline employment. We solve its clearing equations directly; no argument below takes $\kappa_g \downarrow 1$ in group earnings, where $\Gamma(1 - 1/\kappa_g)$ diverges.

Standing assumptions. The sets of sectors and worker groups are finite. Sector weights satisfy $\beta_s > 0$ and $\sum_s \beta_s = 1$. The primitives $M, T_{Ls}, T_{Ms,0}, \gamma_s$ and ψ are finite and strictly positive. Technology follows $T_{Ms}(A) = T_{Ms,0}A^{\nu_s}$ with $A \geq 1$ and $\nu_s \geq 0$; the long-run growth results additionally assume $\bar{\nu} \equiv \sum_s \beta_s \nu_s > 0$. A mobile worker-profile collection is admissible if every group has finite mass $\bar{L}_g > 0$, shape $1 < \kappa_g < \infty$, and finite access parameters $a_{gs} \geq 0$; every group can work in at least one sector, and every sector is supported by at least one positive-mass group. We represent each group's Fréchet law on the canonical type space in which $0 < z_s < \infty$ whenever $a_{gs} > 0$, while $z_s = 0$ whenever $a_{gs} = 0$. An admissible specific-factors structure has $0 < \ell_s^0 < \infty$ in every sector. Specific-factor workers likewise have fixed, finite, strictly positive efficiency units in their assigned sector. These assumptions make all sums finite and ensure that every minimum over groups or sectors is attained.

A. Sufficient statistics for group real income and output. For any $x > 0$, the Roy distribution gives

$$\Pr\left[\max_s \{w_s z_s\} \leq x\right] = \exp\left[-x^{-\kappa_g} \sum_s a_{gs} w_s^{\kappa_g}\right].$$

The group's average earnings and sectoral employment shares are therefore

$$W_g = \Gamma\left(1 - \frac{1}{\kappa_g}\right) \left(\sum_s a_{gs} w_s^{\kappa_g}\right)^{1/\kappa_g}, \tag{B146}$$

$$\mu_{gs} = \frac{a_{gs} w_s^{\kappa_g}}{\sum_k a_{gk} w_k^{\kappa_g}}, \tag{B147}$$

where $a_{gs} = 0$ means that group g cannot work in sector s and therefore implies $\mu_{gs} = 0$. Differentiating the first identity and using Euler's theorem yields

$$\frac{\partial \ln W_g}{\partial \ln w_s} = \mu_{gs}, \qquad \sum_s w_s \frac{\partial W_g}{\partial w_s} = W_g. \tag{B148}$$

Hence equilibrium efficiency-unit employment in sector s is $\ell_s = \sum_g \bar{L}_g \mu_{gs} W_g / w_s$, and its wage bill is $w_s \ell_s = \sum_g \bar{L}_g \mu_{gs} W_g$.

The within-sector task technology determines labor's and AI's shares of task spending and the sector price:

$$\pi_{Ls} = \frac{T_{Ls}}{T_{Ls} + T_{Ms}(w_s/r)^{\psi}}, \qquad \pi_{Ms} = 1 - \pi_{Ls} = \frac{T_{Ms}(w_s/r)^{\psi}}{T_{Ls} + T_{Ms}(w_s/r)^{\psi}},$$
$$P_s = \gamma_s w_s \left[T_{Ls} + T_{Ms}(w_s/r)^{\psi}\right]^{-1/\psi} = \gamma_s w_s \left(\frac{\pi_{Ls}}{T_{Ls}}\right)^{1/\psi}. \tag{B149}$$

The last equality for P_s follows by substituting the expression for π_{Ls} into the sectoral unit cost. Sectoral labor-market clearing gives

$$w_s \ell_s = \beta_s \pi_{Ls} Y. \tag{B150}$$

Dividing by ℓ_s isolates the sector real wage:

$$w_s = Y \frac{\beta_s \pi_{Ls}}{\ell_s}. \tag{B151}$$

Equation (B151) writes the sector real wage as aggregate output times labor's share of sectoral spending per efficiency unit employed. Substituting this expression into equation (B146) gives the group-level sufficient-statistics formula:

$$W_g = \Gamma\left(1 - \frac{1}{\kappa_g}\right) Y \left[\sum_s a_{gs} \left(\frac{\beta_s \pi_{Ls}}{\ell_s}\right)^{\kappa_g}\right]^{1/\kappa_g}. \tag{B152}$$

Thus group g's average real income equals aggregate output times the constant that converts the Fréchet scale into mean earnings and the group-specific expression in brackets. The expression in brackets depends on the sectors the group can enter, labor's task share in those sectors, and the equilibrium allocation of efficiency units across sectors.

To recover output, substitute the sector-price expression in equation (B149) and labor-market clearing into the price normalization $1 = \prod_k P_k^{\beta_k}$:

$$
\begin{aligned}
1 &= \prod_k \left[\frac{\gamma_k \beta_k Y \pi_{Lk}^{1+1/\psi}}{T_{Lk}^{1/\psi} \ell_k} \right]^{\beta_k} \\
&= Y \prod_k \left[\frac{\gamma_k \beta_k \pi_{Lk}^{1+1/\psi}}{T_{Lk}^{1/\psi} \ell_k} \right]^{\beta_k} .
\end{aligned}
$$

The second line uses $\sum_k \beta_k = 1$. Solving for Y gives the output sufficient-statistics formula:

$$
Y = \prod_k \left[\frac{T_{Lk}^{1/\psi} \ell_k}{\gamma_k \beta_k \pi_{Lk}^{1+1/\psi}} \right]^{\beta_k} . \tag{B153}
$$

Together, equations (B152) and (B153) map the vectors $\{\pi_{Ls}\}_s$ and $\{\ell_s\}_s$ into aggregate output and each group's average real income.

Exact changes. Let a hat denote the ratio of a counterfactual value to its baseline value. The counterfactual changes AI technology while holding T_{Lk}, γ_k, and β_k fixed. It also holds the worker-group parameters a_{gk} and κ_g fixed. Taking the ratio of equation (B153) across the two equilibria gives

$$
\hat{Y} = \prod_k \left[\frac{\hat{\ell}_k}{\hat{\pi}_{Lk}^{1+1/\psi}} \right]^{\beta_k} . \tag{B154}
$$

Thus the change in output depends on changes in labor's task shares and the allocation of efficiency units across sectors.

For group earnings, the baseline weight on sector s in the Fréchet power mean is its baseline employment share:

$$
\frac{a_{gs}(\beta_s \pi_{Ls}/\ell_s)^{\kappa_g}}{\sum_k a_{gk}(\beta_k \pi_{Lk}/\ell_k)^{\kappa_g}} = \frac{a_{gs} w_s^{\kappa_g}}{\sum_k a_{gk} w_k^{\kappa_g}} = \mu_{gs}.
$$

The first equality follows from equation (B151) because the common factor Y cancels from the weights. Taking the ratio of equation (B146) therefore gives the change in average earnings,

$$
\hat{W}_g = \left[\sum_s \mu_{gs} \hat{w}_s^{\kappa_g} \right]^{1/\kappa_g} .
$$

Average earnings therefore change as a Fréchet power mean of sector wage changes, with baseline sectoral employment shares as weights. Equivalently, taking the ratio of equation (B152) gives the sufficient-statistics expression for the change in average real income:

$$\hat{W}_g = \hat{Y}\left[\sum_s \mu_{gs}\left(\frac{\hat{\pi}_{Ls}}{\hat{\ell}_s}\right)^{\kappa_g}\right]^{1/\kappa_g}. \tag{B155}$$

This expression separates the common change in aggregate output from the group-specific effect of changes in labor's task share per efficiency unit, weighted by baseline sectoral employment shares. Here μ_{gs} is the baseline employment share defined in equation (B147).

Summing labor income across sectors and imposing AI-market clearing gives

$$\frac{Y}{r} = M + \sum_g \bar{L}_g \frac{W_g}{r}, \qquad M = \frac{Y}{r}\sum_s \beta_s \pi_{Ms}. \tag{B156}$$

The second equality is AI-market clearing. Combining it with equation (B150), summed across sectors, gives the first equality, the aggregate income identity. We now use these identities to establish the bounds required for the mobility-floor lemma and Theorem 4.

B. Equilibrium existence and uniqueness. We prove existence and uniqueness by expressing market clearing as the first-order conditions of a single minimization problem. For a fixed technology vector, define the following function of the log wage-rental ratios:

$$\begin{aligned}\mathcal{V}\left(\left\{\ln\frac{w_s}{r}\right\}_s\right) = \ln&\left[M + \sum_g \bar{L}_g \Gamma\left(1-\frac{1}{\kappa_g}\right)\left(\sum_s a_{gs}(w_s/r)^{\kappa_g}\right)^{1/\kappa_g}\right] \\ &- \sum_s \beta_s \ln\left\{\gamma_s \frac{w_s}{r}\left[T_{Ls} + T_{Ms}(w_s/r)^{\psi}\right]^{-1/\psi}\right\}. \end{aligned}\tag{B157}$$

We first show that $\mathcal{V}$ is strictly convex. Writing $x_s = \ln(w_s/r)$, each group term has a logarithm of the form

$$\frac{1}{\kappa_g}\ln\left(\sum_s a_{gs}e^{\kappa_g x_s}\right),$$

which is convex in the log wage-rental ratios. Hölder's inequality preserves log-convexity when the finitely many group terms and the constant M are added. The sector-price part contributes the strictly positive diagonal term

$$\frac{\partial^2 \mathcal{V}}{\partial[\ln(w_s/r)]^2} \supset \beta_s \psi \pi_{Ms}(1-\pi_{Ms}) > 0.$$

Hence $\mathcal{V}$ is strictly convex.

We next show that $\mathcal{V}$ tends to infinity whenever the vector of log wage-rental ratios leaves every bounded set. This property ensures that the objective attains a minimum. Every sector appears in at least one group's support because it has positive baseline employment. The first line of (B157) therefore controls the largest log wage-rental ratio, while the sector-price terms control every ratio that becomes negative. Up to an additive constant,

$$\mathcal{V}\left(\left\{\ln\frac{w_s}{r}\right\}_s\right) \geq \left[\max_s \ln\frac{w_s}{r}\right]_+ + \sum_s \beta_s \left[-\ln\frac{w_s}{r}\right]_+ .$$

The right-hand side diverges whenever the vector $\{\ln(w_s/r)\}_s$ becomes unbounded: either its largest component tends to infinity, or some component tends to minus infinity while its largest component remains bounded. Thus $\mathcal{V}$ is coercive.

The derivative with respect to $\ln(w_s/r)$ is

$$\frac{\sum_g \bar{L}_g \mu_{gs}(W_g/r)}{M + \sum_g \bar{L}_g(W_g/r)} - \beta_s(1 - \pi_{Ms}).$$

The first-order conditions are therefore exactly (B150). Coercivity gives existence, and strict convexity gives uniqueness. In the specific-factors economy, replacing group earnings by $M + \sum_s \ell_s^0(w_s/r)$ leaves the strictly positive sector-price curvature unchanged. Because every ℓ_s^0 is positive, the objective also tends to infinity when any log wage-rental ratio becomes unbounded. The specific-factors equilibrium is therefore also unique and interior. Interiority in supported activities is a conclusion, not an assumption.

C. A uniform bound on sectoral labor supply. For $a_{gs} > 0$,

$$\begin{aligned}\frac{\partial W_g}{\partial w_s} &= \Gamma\left(1 - \frac{1}{\kappa_g}\right) a_{gs} w_s^{\kappa_g - 1} \left(\sum_k a_{gk} w_k^{\kappa_g}\right)^{1/\kappa_g - 1} \\ &\leq \Gamma\left(1 - \frac{1}{\kappa_g}\right) a_{gs} w_s^{\kappa_g - 1} \left(a_{gs} w_s^{\kappa_g}\right)^{1/\kappa_g - 1} = \Gamma\left(1 - \frac{1}{\kappa_g}\right) a_{gs}^{1/\kappa_g}.\end{aligned}$$

The inequality has this direction because $1/\kappa_g - 1 < 0$, so raising $\sum_k a_{gk} w_k^{\kappa_g} \geq a_{gs} w_s^{\kappa_g}$ to that power reverses the order. When $a_{gs} = 0$, the derivative is zero. Consequently,

$$\ell_s \leq \sum_g \bar{L}_g \Gamma\left(1 - \frac{1}{\kappa_g}\right) a_{gs}^{1/\kappa_g} < \infty. \tag{B158}$$

This bound applies under sparse support and uses efficiency units rather than headcounts.

D. Aggregate income bounds. Equation (B156) gives $Y/r \geq M$. It also gives a bound that is uniform in A. Suppose, to the contrary, that a sequence of equilibria has $Y/r \to \infty$. Since the number of sectors is finite and every β_s is positive, the identity $M/(Y/r) = \sum_s \beta_s \pi_{Ms}$ implies

$\pi_{Ms} \to 0$ in every sector. Equations (B150) and (B158) then imply

$$\frac{w_s}{r} \geq \frac{\beta_s(1-\pi_{Ms})(Y/r)}{\sum_g \bar{L}_g \Gamma(1-1/\kappa_g) a_{gs}^{1/\kappa_g}} \longrightarrow \infty.$$

But $T_{Ms}(A) \geq T_{Ms,0} > 0$ for $A \geq 1$, so (B149) instead implies $\pi_{Ms} \to 1$. In the specific-factors economy, the same contradiction follows directly from fixed $\ell_s^0 > 0$ in place of the mobile labor bound. The contradiction proves

$$M \leq \frac{Y}{r} \leq \sup_{\widetilde{A} \geq 1} \frac{Y(\widetilde{A})}{r(\widetilde{A})} < \infty, \qquad M \leq \frac{Y}{r}\bigg|_{\kappa \downarrow 1} \leq \sup_{\widetilde{A} \geq 1} \frac{Y(\widetilde{A})}{r(\widetilde{A})}\bigg|_{\kappa \downarrow 1} < \infty \quad \text{for } A \geq 1. \tag{B159}$$

E. The mobility floor. The next lemma combines the labor-supply and aggregate-income bounds into the formal result used in Section 5.

Lemma B.1 (Mobility floor)**.** *For every admissible collection of worker-group Fréchet distributions and every sector s, there is a constant $\underline{c}_s > 0$, independent of A, such that*

$$w_s \geq \underline{c}_s \, w_s|_{\kappa \downarrow 1} \qquad \textit{for all } A \geq 1.$$

Moreover, there is a constant $\underline{d}_s > 0$, independent of A, such that

$$w_s \geq \underline{d}_s A^{(\sum_k \beta_k \nu_k)/\psi - \nu_s/(1+\psi)} \qquad \textit{for all } A \geq 1,$$

whereas

$$w_s|_{\kappa \downarrow 1} \asymp A^{(\sum_k \beta_k \nu_k)/\psi - \nu_s/(1+\psi)} \qquad \textit{as } A \to \infty.$$

In words, worker sorting cannot reduce a sector's real wage below a fixed multiple of its specific-factors counterpart. The benchmark identifies its own sectoral growth rate, while the mobile economy is guaranteed to grow at least as fast in exponent terms. Thus specific factors provide a lower rate benchmark and, up to a fixed multiplicative constant, a uniform level floor for sector wages in the mobile economy.

Proof. The proof has two steps. Step 1 compares sectoral wage-rental ratios in the mobile and specific-factors economies. Step 2 uses the price normalization to convert those comparisons into bounds on real wages and output.

Step 1. Sector wages. Substituting the task share into (B150) and applying (B158) together with (B159) gives, in the mobile economy,

$$\frac{w_s}{r} + \frac{T_{Ms,0}}{T_{Ls}} A^{\nu_s} \left(\frac{w_s}{r}\right)^{1+\psi} \geq \frac{\beta_s M}{\sum_g \bar{L}_g \Gamma(1-1/\kappa_g) a_{gs}^{1/\kappa_g}}.$$

Denote the fixed positive right-hand side by C_s. Since both terms on the left are nonnegative, at least one is no smaller than $C_s/2$. If the linear term is no smaller than $C_s/2$, then $w_s/r \geq C_s/2$. Otherwise, the nonlinear term is no smaller than $C_s/2$, which gives a positive multiple of $A^{-\nu_s/(1+\psi)}$. Since $A \geq 1$, the two cases combine to give the direct bound

$$\frac{w_s}{r} \geq \min\left\{ \frac{\beta_s M}{2\sum_g \bar{L}_g \Gamma(1-1/\kappa_g) a_{gs}^{1/\kappa_g}}, \left[\frac{\beta_s M T_{Ls}}{2T_{Ms,0} \sum_g \bar{L}_g \Gamma(1-1/\kappa_g) a_{gs}^{1/\kappa_g}} \right]^{1/(1+\psi)} \right\} A^{-\nu_s/(1+\psi)}. \tag{B160}$$

Efficiency-unit employment is fixed at $\ell_s^0 > 0$ in the specific-factors economy. Its clearing equation can be written directly as

$$\left[\frac{w_s}{r} + \frac{T_{Ms,0}}{T_{Ls}} A^{\nu_s} \left(\frac{w_s}{r} \right)^{1+\psi} \right]\Bigg|_{\kappa\downarrow 1} = \frac{\beta_s}{\ell_s^0} \left. \frac{Y}{r} \right|_{\kappa\downarrow 1}.$$

In the specific-factors equation, the lower aggregate-income bound makes the right-hand side at least $\beta_s M/\ell_s^0$. Splitting the two positive terms at one-half therefore gives the same lower rate, a positive multiple of $A^{-\nu_s/(1+\psi)}$ as a lower bound. Discarding the positive linear term and using the upper income bound gives

$$\left(\frac{w_s}{r} \right)^{1+\psi} \Bigg|_{\kappa\downarrow 1} \leq \frac{\beta_s T_{Ls}}{\ell_s^0 T_{Ms,0}} \sup_{\widetilde{A}\geq 1} \left. \frac{Y(\widetilde{A})}{r(\widetilde{A})} \right|_{\kappa\downarrow 1} A^{-\nu_s}.$$

Together with (B160), this proves

$$\left. \frac{w_s}{r} \right|_{\kappa\downarrow 1} \asymp A^{-\nu_s/(1+\psi)} \quad \text{and} \quad \frac{w_s}{r} \geq c_s \left. \frac{w_s}{r} \right|_{\kappa\downarrow 1} \tag{B161}$$

for a constant $c_s > 0$ independent of A.

Step 2. Prices and real wages. Equation (B149) implies

$$\begin{aligned} r &= \prod_s \left[\gamma_s^{-1} \left(T_{Ms,0} A^{\nu_s} + T_{Ls} (w_s/r)^{-\psi} \right)^{1/\psi} \right]^{\beta_s}, \\ r|_{\kappa\downarrow 1} &= \prod_s \left[\gamma_s^{-1} \left(T_{Ms,0} A^{\nu_s} + T_{Ls} (w_s/r)^{-\psi} \right)^{1/\psi} \right]^{\beta_s} \Bigg|_{\kappa\downarrow 1}. \end{aligned} \tag{B162}$$

Dropping the labor term inside every parenthesis gives a fixed positive multiple of $A^{(\sum_s \beta_s \nu_s)/\psi}$ as a lower bound in both economies. The sector-wage bounds make each labor term no larger than a fixed positive multiple of $A^{\psi\nu_s/(1+\psi)}$, and hence no larger than a fixed positive multiple of A^{ν_s}. Thus every bracket in (B162) is bounded above and below by positive multiples of A^{ν_s}. Taking the expenditure-weighted product gives the stated rate for r. Finally, $Y = r(Y/r)$, and the aggregate-income bounds in equation (B159) bound the second factor above and below by positive constants. Output therefore has the same asymptotic rate as r:

$$r \asymp A^{(\sum_s \beta_s \nu_s)/\psi}, \qquad r|_{\kappa\downarrow 1} \asymp A^{(\sum_s \beta_s \nu_s)/\psi}, \qquad Y \asymp A^{(\sum_s \beta_s \nu_s)/\psi}, \qquad Y|_{\kappa\downarrow 1} \asymp A^{(\sum_s \beta_s \nu_s)/\psi}. \tag{B163}$$

Each real wage equals its wage-rental ratio times its economy's AI rental. Equations (B161) and (B163) first imply a constant $\underline{d}_s > 0$ such that

$$w_s \geq \underline{d}_s A^{(\sum_k \beta_k \nu_k)/\psi - \nu_s/(1+\psi)}, \qquad w_s|_{\kappa\downarrow 1} \asymp A^{(\sum_k \beta_k \nu_k)/\psi - \nu_s/(1+\psi)}.$$

The same bounds imply a constant $\underline{c}_s > 0$ such that

$$w_s \geq \underline{c}_s \, w_s|_{\kappa\downarrow 1}$$

for every $A \geq 1$. This completes the proof. □

F. Proof of Theorem 4. The quantifier in the phrase "regardless of workers' profiles" applies to the entire admissible class. Let $\bar{\nu} \equiv \sum_s \beta_s \nu_s > 0$ and $\nu_{\max} \equiv \max_s \nu_s$. The theorem makes two claims. First, it gives each sector's limiting real-wage growth share in the specific-factors benchmark:

$$\left.\frac{g_{w_s}}{g_Y}\right|_{\kappa\downarrow 1} \longrightarrow 1 - \frac{\psi}{1+\psi}\frac{\nu_s}{\bar{\nu}}.$$

Second, it states that

$$\frac{\nu_{\max}}{\bar{\nu}} < \frac{1+\psi}{\psi}$$

if and only if every worker's real income diverges in every admissible worker-market structure. This equivalence is uniform over admissible structures; it does not assert that the condition is necessary within each particular mobile-worker economy.

Proof. We proceed in three steps. Step 1 derives sectoral real-wage growth in the specific-factors benchmark. Step 2 uses the mobility-floor lemma to prove sufficiency for every admissible worker profile. Step 3 proves sharpness over the admissible class by placing workers in the sector with the fastest AI-productivity growth. *Step 1. Specific-factors growth rates.* Strict convexity and the implicit-function theorem make the specific-factors equilibrium path differentiable. Differentiating sectoral labor-market clearing gives

$$\left.\left[(1+\psi\pi_{Ms})\,\frac{\mathrm{d}\ln(w_s/r)}{\mathrm{d}\ln A}\right]\right|_{\kappa\downarrow 1} = \left.\left[\frac{\mathrm{d}\ln(Y/r)}{\mathrm{d}\ln A} - \nu_s\pi_{Ms}\right]\right|_{\kappa\downarrow 1}.$$

Using the income identity to substitute for the derivative of $\frac{Y}{r}\big|_{\kappa\downarrow 1}$ yields

$$\left\{\frac{\mathrm{d}\ln(Y/r)}{\mathrm{d}\ln A}\left[1-\sum_s \frac{\ell_s^0(w_s/r)}{(Y/r)(1+\psi\pi_{Ms})}\right]\right\}\Bigg|_{\kappa\downarrow 1}$$
$$= -\sum_s \frac{\ell_s^0(w_s/r)\nu_s\pi_{Ms}}{(Y/r)(1+\psi\pi_{Ms})}\Bigg|_{\kappa\downarrow 1}.$$

The term in brackets is at least $\frac{M}{Y/r}\big|_{\kappa\downarrow 1}$, which is uniformly positive by the income bound. Equation (B161) implies that $\frac{w_s}{r}\big|_{\kappa\downarrow 1} \to 0$ and $\pi_{Ms}|_{\kappa\downarrow 1} \to 1$ whenever $\nu_s > 0$; sectors with $\nu_s = 0$ contribute zero to the right-hand side. Hence $\frac{Y}{r}\big|_{\kappa\downarrow 1}$ has asymptotically zero log derivative, and the differentiated clearing equation gives

$$\frac{\mathrm{d}\ln(w_s/r)}{\mathrm{d}\ln A}\bigg|_{\kappa\downarrow 1} \longrightarrow -\frac{\nu_s}{1+\psi} \qquad \text{for every } s.$$

The labor-share formula in equation (B149) gives

$$\frac{\mathrm{d}\ln\pi_{Ls}}{\mathrm{d}\ln A}\bigg|_{\kappa\downarrow 1} = -\,\pi_{Ms}\left[\nu_s + \psi\frac{\mathrm{d}\ln(w_s/r)}{\mathrm{d}\ln A}\right]\bigg|_{\kappa\downarrow 1} \longrightarrow -\frac{\nu_s}{1+\psi}.$$

Efficiency-unit employment is fixed in the specific-factors economy. Differentiating the output sufficient-statistics formula in equation (B153) therefore gives

$$\frac{\mathrm{d}\ln Y}{\mathrm{d}\ln A}\bigg|_{\kappa\downarrow 1} = -\left(1+\frac{1}{\psi}\right)\sum_k \beta_k \frac{\mathrm{d}\ln\pi_{Lk}}{\mathrm{d}\ln A}\bigg|_{\kappa\downarrow 1} \longrightarrow \frac{\sum_k \beta_k\nu_k}{\psi}.$$

Finally, differentiating the sector real wage formula in equation (B151) yields

$$\frac{\mathrm{d}\ln w_s}{\mathrm{d}\ln A}\bigg|_{\kappa\downarrow 1} = \frac{\mathrm{d}\ln Y}{\mathrm{d}\ln A}\bigg|_{\kappa\downarrow 1} + \frac{\mathrm{d}\ln\pi_{Ls}}{\mathrm{d}\ln A}\bigg|_{\kappa\downarrow 1} \longrightarrow \frac{\sum_k \beta_k\nu_k}{\psi} - \frac{\nu_s}{1+\psi}.$$

This expression separates a common and a sector-specific force. The term $\bar{\nu}/\psi$ is the contribution of aggregate output growth, which is common to all sectors. The term $\nu_s/(1+\psi)$ captures the decline in labor's task share in sector s. A sector's real wage diverges when the common output effect exceeds this sector-specific displacement effect. Along any differentiable path with $g_A > 0$, the chain rule gives the theorem's growth-share statement.

Step 2. Sufficiency for every admissible structure. Suppose first that

$$\frac{\nu_{\max}}{\bar{\nu}} < \frac{1+\psi}{\psi}.$$

For every sector s,

$$\frac{\bar{\nu}}{\psi} - \frac{\nu_s}{1+\psi} \geq \frac{\bar{\nu}}{\psi} - \frac{\nu_{\max}}{1+\psi} > 0.$$

The mobility-floor lemma therefore implies $w_s \to \infty$ in every admissible mobile economy and in the specific-factors benchmark. For a mobile group g, choose any sector s with $a_{gs} > 0$. Equation (B146) gives

$$W_g \geq \Gamma\left(1 - \frac{1}{\kappa_g}\right) a_{gs}^{1/\kappa_g} w_s,$$

so every group's average real income diverges. A realized worker with productivity vector z earns at least

$$\max_{k: a_{gk} > 0} z_k w_k \geq z_s w_s.$$

By the canonical type-space convention, the chosen supported sector satisfies $0 < z_s < \infty$ for every worker in group g. Since $w_s \to \infty$, every worker's real income diverges.

Step 3. Necessity for a uniform guarantee. For the converse, suppose $\nu_{\max}/\bar{\nu} \geq (1+\psi)/\psi$ and choose a sector s^* with $\nu_{s^*} = \nu_{\max}$. The admissible specific-factors benchmark includes workers fixed in sector s^* and satisfies

$$w_{s^*}|_{\kappa \downarrow 1} \asymp A^{\bar{\nu}/\psi - \nu_{\max}/(1+\psi)}.$$

At equality this real wage remains bounded above and below by positive constants; under the strict reverse inequality it converges to zero. Hence not every worker's real income diverges in every admissible structure. When the condition fails, the fastest-growing sector therefore provides an admissible worker attachment for which real income does not diverge. The condition is thus necessary for a guarantee that is uniform over all admissible worker structures. Together with the specific-factors growth formula and the sufficiency argument, this establishes Theorem 4. □

G. Further implications and full support.

Remark (Groups with access to slower sectors). The same lower bound implies that group g's average real income satisfies the following lower bound on its long-run growth exponent relative to output:

$$1 - \frac{\psi}{1+\psi} \frac{\min\{\nu_s : a_{gs} > 0\}}{\sum_k \beta_k \nu_k}.$$

Under full support, the minimum in the numerator is $\min_s \nu_s$ for every group. If $\min_s \nu_s = 0$, the exponent of the group's average real income is at least $(\sum_k \beta_k \nu_k)/\psi > 0$. If $\min_s \nu_s > 0$, then $\sum_k \beta_k \nu_k \geq \min_s \nu_s$ and

$$\frac{\sum_k \beta_k \nu_k}{\psi} - \frac{\min_s \nu_s}{1+\psi} \geq \frac{\min_s \nu_s}{\psi(1+\psi)} > 0.$$

A permanently attached worker in a sector with exponent $\min\{\nu_s : a_{gs} > 0\}$ attains the corresponding growth share.

B.8 Elastic final demand and the Jevons threshold

This subsection derives the results of Section 5.1 in the economy of Section 5, without intermediate inputs and without sector-specific capital. Primitives and standing assumptions are those of Appendix B.7; only the final demand aggregator changes, to the one in Appendix B.6. Hats denote proportional changes, $\hat{x} = x'/x$, and time subscripts are dropped. Part G states the version of this economy behind Figure 10, which keeps intermediate inputs and sector capital.

A. Elastic final demand and market clearing. Let $\eta > 0$ be the elasticity of substitution across sectors and replace the Cobb–Douglas aggregator by

$$Y = \left[\sum_s \beta_s^{1/\eta} Y_s^{(\eta-1)/\eta}\right]^{\eta/(\eta-1)}, \qquad 1 = \left[\sum_s \beta_s P_s^{1-\eta}\right]^{1/(1-\eta)}. \tag{B164}$$

The expenditure share $\omega_s \equiv P_s Y_s / Y$ is then (32), and the constant β_s at $\eta = 1$. Units are chosen so that every baseline sector price equals one, making β_s the baseline expenditure share as well. Sector technology is unchanged:

$$\begin{aligned}
\pi_{Ls} &= \frac{T_{Ls}}{T_{Ls} + T_{Ms}(w_s/r)^{\psi}}, \qquad \pi_{Ms} = 1 - \pi_{Ls}, \\
P_s &= \gamma_s w_s \left[T_{Ls} + T_{Ms}(w_s/r)^{\psi}\right]^{-1/\psi} \\
&= \gamma_s \left[T_{Ls} w_s^{-\psi} + T_{Ms} r^{-\psi}\right]^{-1/\psi} = \gamma_s w_s \left(\frac{\pi_{Ls}}{T_{Ls}}\right)^{1/\psi}.
\end{aligned} \tag{B165}$$

The middle form is a CES unit cost in the two factor prices with elasticity of substitution $1 + \psi$. Labor market clearing in sector s reads

$$\begin{aligned}
w_s \ell_s &= \sum_g \bar{L}_g \mu_{gs} W_g = \omega_s \pi_{Ls} Y, \\
\mu_{gs} &= \frac{a_{gs} w_s^{\kappa_g}}{\sum_k a_{gk} w_k^{\kappa_g}}, \qquad W_g = \Gamma\left(1 - \frac{1}{\kappa_g}\right)\left(\sum_s a_{gs} w_s^{\kappa_g}\right)^{1/\kappa_g},
\end{aligned} \tag{B166}$$

with the employment shares and group earnings of Section 5 in the second line. Income is paid to the two factors, and summing (B166) across sectors gives AI market clearing:

$$Y = rM + \sum_g \bar{L}_g W_g, \qquad \sum_s \omega_s \pi_{Ms} Y = rM. \tag{B167}$$

These S conditions in the S unknowns w_s/r are the full system; the specific-factors economy replaces Roy supply by fixed efficiency units $\ell_s = \ell_s^0$, as in Appendix B.7. With $Z_s \equiv \ln[T_{Ls} + T_{Ms}(w_s/r)^{\psi}]$, the log derivatives used below are

$$\begin{aligned} \mathrm{d}Z_s &= \pi_{Ms}\left[\mathrm{d}\ln T_{Ms} + \psi\,\mathrm{d}\ln(w_s/r)\right], & \mathrm{d}\ln P_s &= \mathrm{d}\ln w_s - \frac{1}{\psi}\,\mathrm{d}Z_s, \\ \mathrm{d}\ln \pi_{Ls} &= -\,\mathrm{d}Z_s, & \mathrm{d}\ln \omega_s &= (1-\eta)\left[\mathrm{d}\ln P_s - \sum_k \omega_k\,\mathrm{d}\ln P_k\right]. \end{aligned} \tag{B168}$$

B. Existence and uniqueness for $\eta \geq 1$. Write $x_s \equiv \ln(w_s/r)$ and define the potential

$$\mathcal{V}_\eta(x) = \ln\left[M + \sum_g \bar{L}_g \frac{W_g}{r}\right] - \ln\frac{P}{r}, \tag{B169}$$

with $W_g/r = \Gamma(1-1/\kappa_g)(\sum_s a_{gs}e^{\kappa_g x_s})^{1/\kappa_g}$, $P_s/r = \gamma_s e^{x_s}[T_{Ls} + T_{Ms}e^{\psi x_s}]^{-1/\psi}$ and $\ln(P/r) = (1-\eta)^{-1}\ln\sum_s \beta_s(P_s/r)^{1-\eta}$, read as $\sum_s \beta_s \ln(P_s/r)$ at $\eta = 1$, where it is the potential of Appendix B.7. Its gradient is

$$\frac{\partial \mathcal{V}_\eta}{\partial x_s} = \frac{\sum_g \bar{L}_g \mu_{gs} W_g}{rM + \sum_g \bar{L}_g W_g} - \omega_s \pi_{Ls} = \frac{w_s \ell_s - \omega_s \pi_{Ls} Y}{Y}, \tag{B170}$$

so its critical points are exactly the equilibria (B166). The first term is convex in x, as shown in Appendix B.7; the Hessian of the second is

$$-\frac{\partial^2 \ln(P/r)}{\partial x_s \partial x_k} = \psi\delta_{sk}\omega_s\pi_{Ls}\pi_{Ms} + (\eta-1)\left[\delta_{sk}\omega_s\pi_{Ls}^2 - \omega_s\pi_{Ls}\omega_k\pi_{Lk}\right], \tag{B171}$$

whose bracketed matrix is positive semidefinite, since for every vector z

$$\sum_s \omega_s \pi_{Ls}^2 z_s^2 - \left(\sum_s \omega_s \pi_{Ls} z_s\right)^2 \geq 0$$

by the Cauchy–Schwarz inequality and $\sum_s \omega_s = 1$. Hence $\mathcal{V}_\eta$ is strictly convex for every $\eta \geq 1$. It is also coercive: up to additive constants, the first term is at least $[\max_s x_s]_+$, as shown there, and for $\eta > 1$

$$-\ln\frac{P}{r} \geq \max_s\left[-\ln\frac{P_s}{r}\right] + \frac{1}{\eta-1}\min_s \ln\beta_s \geq \max_s\,[-x_s]_+,$$

so $\mathcal{V}_\eta$ diverges whenever x leaves every bounded set. The equilibrium therefore exists and is unique for every $\eta \geq 1$, in the Roy economy and in the specific-factors economy, where $M + \sum_s \ell_s^0 e^{x_s}$ replaces the group term. Parts C and D use no uniqueness.

C. Labor demand in one sector and the threshold. Let AI advance in sector s alone, in a sector small enough that Y, r and the other sectors' wages are unaffected, which is exact as $\beta_s \to 0$. The wage bill demanded at w_s is $D_s(w_s) = \omega_s \pi_{Ls} Y$, so (B168) gives

$$\mathrm{d}\ln D_s = \frac{\eta - 1 - \psi}{\psi} \pi_{Ms} \,\mathrm{d}\ln T_{Ms} - \left[(\eta - 1)\pi_{Ls} + \psi \pi_{Ms}\right] \mathrm{d}\ln w_s. \tag{B172}$$

The first coefficient sets two forces against each other: AI progress lowers the sector price by $1/\psi$ times $\mathrm{d}Z_s$, which raises the expenditure share by $(\eta - 1)/\psi$ times $\mathrm{d}Z_s$, while the labor task share falls by the full $\mathrm{d}Z_s$. Labor demand $\ell_s^d = D_s/w_s$ has wage elasticity $-[\eta\pi_{Ls} + (1 + \psi)\pi_{Ms}] < 0$ for every $\eta > 0$, and Roy labor supply $\ell_s = \sum_g \bar{L}_g \mu_{gs} W_g / w_s$ has wage elasticity

$$\varepsilon_s \equiv \frac{\mathrm{d}\ln \ell_s}{\mathrm{d}\ln w_s} = \sum_g \theta_{gs}(\kappa_g - 1)(1 - \mu_{gs}) \geq 0, \qquad \theta_{gs} \equiv \frac{\bar{L}_g \mu_{gs} W_g}{w_s \ell_s}, \tag{B173}$$

with θ_{gs} group g's share of the sector's wage bill and $\varepsilon_s = 0$ in the specific-factors economy. Equating demand and supply,

$$\frac{\mathrm{d}\ln w_s}{\mathrm{d}\ln T_{Ms}} = \frac{(\eta - 1 - \psi)\pi_{Ms}}{\psi\left[\eta\pi_{Ls} + (1 + \psi)\pi_{Ms} + \varepsilon_s\right]}, \tag{B174}$$

and $\mathrm{d}\ln(w_s \ell_s) = (1 + \varepsilon_s)\,\mathrm{d}\ln w_s$. The denominator is positive, so the sector's wage and wage bill rise with its own AI progress if and only if $\eta > 1 + \psi$, for any $\eta > 0$ and any κ_g: the threshold of Section 5.1.

D. The specific-factors economy. With $\ell_s = \ell_s^0$, divide (B166) for sector s by the same condition for sector k and take proportional changes. Since $\hat{\omega}_s / \hat{\omega}_k = (\hat{P}_s / \hat{P}_k)^{1-\eta}$,

$$\frac{\hat{w}_s}{\hat{w}_k} = \left(\frac{\hat{P}_s}{\hat{P}_k}\right)^{1-\eta} \frac{\hat{\pi}_{Ls}}{\hat{\pi}_{Lk}}, \tag{B175}$$

and substituting $\hat{P}_s = \hat{w}_s \hat{\pi}_{Ls}^{1/\psi}$ from (B165) gives (33), which holds exactly for any pattern of AI progress. Equivalently $\hat{w}_s^{\eta} = C \hat{\pi}_{Ls}^{(1+\psi-\eta)/\psi}$ with C common to all sectors: below the threshold the sector whose labor task share falls most has the lowest wage growth, above it the highest, and at $\eta = 1 + \psi$ all sector wages change by the same factor.

E. The common factor at $\eta = 1 + \psi$. Now let workers sort with any κ_g; the same argument covers the specific-factors economy. Let $\hat{T}_{Ms} \geq 1$ be arbitrary, write $\hat{q} \equiv \hat{w}/\hat{r}$ for the change in the wage rental ratio, and guess $\hat{w}_s = \hat{w}$ in every sector. Then $\hat{\mu}_{gs} = 1$, $\hat{W}_g = \hat{w}$, $\hat{\ell}_s = 1$, and every sector's wage bill changes by $\hat{w}$. With $B_s \equiv 1/\hat{\pi}_{Ls} = \pi_{Ls} + \pi_{Ms} \hat{T}_{Ms} \hat{q}^{\psi}$ we have $\hat{P}_s = \hat{w} B_s^{-1/\psi}$, and at $1 - \eta = -\psi$ the price normalization and the expenditure share give

$$1 = \hat{P}^{-\psi} = \sum_s \beta_s \hat{P}_s^{-\psi} = \hat{w}^{-\psi} \sum_k \beta_k B_k, \quad \hat{\omega}_s = \hat{P}_s^{-\psi} = \frac{B_s}{\sum_k \beta_k B_k}, \quad \hat{\omega}_s \hat{\pi}_{Ls} = \frac{1}{\sum_k \beta_k B_k},$$

the last of these the same in every sector. Each clearing condition therefore reduces to the single equation

$$\hat{w} \sum_k \beta_k B_k = \hat{Y} = \bar{\pi}_M \hat{r} + (1 - \bar{\pi}_M)\hat{w},$$

where $\bar{\pi}_M \equiv \sum_k \beta_k \pi_{Mk} = rM/Y$ is the baseline share of income paid to AI and the right side is (B167) in proportional changes. Dividing by $\hat{r}$ and using $\sum_k \beta_k \pi_{Lk} = 1 - \bar{\pi}_M$, this is

$$\hat{q}^{1+\psi} = \frac{\sum_k \beta_k \pi_{Mk}}{\sum_k \beta_k \pi_{Mk} \hat{T}_{Mk}}, \tag{B176}$$

with a unique positive solution, which by part B is the equilibrium. Sorting is undisturbed, and every group's real income changes by the common factor

$$\hat{W}_g^{\psi} = \hat{w}^{\psi} = \sum_k \beta_k \left[\pi_{Lk} + \pi_{Mk} \hat{T}_{Mk} \hat{q}^{\psi}\right] = 1 - \bar{\pi}_M + \frac{\bar{\pi}_M}{\hat{q}}. \tag{B177}$$

The pattern of AI progress enters only through the average of $\hat{T}_{Mk}$ weighted by baseline AI income, and every worker's real income rises whenever that average exceeds one.

F. Groups. Taking the ratio of W_g across the two equilibria,

$$\hat{W}_g = \left[\sum_s \mu_{gs} \hat{w}_s^{\kappa_g}\right]^{1/\kappa_g},$$

so a group's earnings change is a power mean of the sector wage changes weighted by its baseline employment shares, and the price index is common to all groups. Combining this with part D, below the threshold a group concentrated in the sectors with the largest fall in the labor task share gains least and above it gains most, and at $\eta = 1 + \psi$ all groups gain the common factor of part E.

G. Intermediate inputs and sector capital. Figure 10 is computed in the calibrated economy of Appendix C with one change: the Cobb–Douglas aggregators over sector outputs become CES aggregators at the elasticity of substitution across sector outputs η, in final and intermediate use alike. Notation is that of Appendix C, the final good is the numeraire, so $\hat{P} = 1$, and the change in the AI rental $\hat{r}$ is determined in equilibrium.

Final expenditure has the price normalization and shares

$$1 = \left[\sum_s \beta_s \hat{P}_s^{1-\eta}\right]^{1/(1-\eta)}, \qquad \omega_s' = \frac{\beta_s \hat{P}_s^{1-\eta}}{\sum_k \beta_k \hat{P}_k^{1-\eta}}. \tag{B178}$$

This replaces equation (C23): expenditure shares move with sector prices and return to β_s at $\eta = 1$.

Each sector's intermediate composite uses the same elasticity, so demand for a sector's output responds to supplier prices as well:

$$\hat{P}^I_s = \left[\sum_k g_{ks}\hat{P}_k^{1-\eta}\right]^{1/(1-\eta)}, \qquad g_{ks} \equiv \frac{\Gamma_{ks}}{1-\alpha_s},$$
$$\Gamma'_{ks} = (1-\alpha_s)\frac{g_{ks}\hat{P}_k^{1-\eta}}{\sum_j g_{js}\hat{P}_j^{1-\eta}},$$
$$\boldsymbol{\lambda}' = (\mathrm{Id} - \Gamma')^{-1}\boldsymbol{\omega}', \qquad \hat{\lambda}_s = \frac{\lambda'_s}{\lambda_s}. \tag{B179}$$

Input shares vary with prices while each column still sums to $1-\alpha_s$, and the Domar weights λ'_s are endogenous instead of fixed.

Sector capital stocks are fixed, so each sector's rental clears its own market and income adjusts with the new weights:

$$\hat{R}_s = \hat{\lambda}_s\hat{I}, \qquad \hat{I} = \frac{M_0\hat{r} + L_0\sum_g \omega_g\hat{W}_g}{1-\sum_k(1-\theta_k)v_k\hat{\lambda}_k}. \tag{B180}$$

Sector s pays its capital $b_s\lambda'_s$ of aggregate income, so its rental follows $\hat{\lambda}_s\hat{I}$ rather than $\hat{I}$, and the capital income share in the denominator replaces the fixed c_K of equation (C26).

The equilibrium solves the sector price equations and the sector worker market clearing conditions:

$$\log\hat{P}_s = b_s\log\hat{R}_s + a_s\log\hat{w}_s - \frac{\theta_s}{\psi}\log\hat{B}_s + (1-\alpha_s)\log\hat{P}^I_s,$$
$$L_0\sum_g \omega_g\hat{W}_g\mu'_{gs} = \hat{I}\theta_s v_s\pi'_{Ls}\hat{\lambda}_s \qquad \text{for every } s. \tag{B181}$$

These replace equations (C22) and (C27), with $\hat{B}_s$, π'_{Ls}, $\hat{W}_g$ and μ'_{gs} unchanged; the extra $\hat{\lambda}_s$ carries the change in sector scale. At $\eta = 1$ the shares and weights return to their baseline values and the system is that of Appendix C.

B.9 Derivations

This section derives the results stated in Section 4.1. We derive the share of tasks done by AI in equation (28), the real wage in equation (29), its change in equation (30), and the limit in equation (31), and we extend that limit to the case in which the share of a task's cost paid to the worker or to the AI differs across sectors. Throughout, the supplies L and M are fixed and positive, both factors move freely across sectors, so that the wage w and the AI rental r are common to all sectors, and we suppress time subscripts where no ambiguity arises.

Environment. Final output aggregates the final uses C_s of the sector outputs with the Cobb–Douglas weights $\beta_s > 0$ of equation (22), so that $1 = \prod_s P_s^{\beta_s}$ and $P_s C_s = \beta_s Y$. Sector output Y_s is now gross output: it serves final demand and the input demand of every sector, $Y_k = C_k + \sum_s X_{ks}$, where X_{ks} is sector s's use of good k. Within sector s, output aggregates a continuum of tasks $i \in [0,1]$ with the constant elasticity $\sigma_s \in (0, 1+\theta)$, and P_s is the corresponding price index, as in Section 2. Each task is produced by workers or by AI. Let $\ell_s(i)$ and $m_s(i)$ denote the worker and AI inputs and $X_s(i)$ the bundle of inputs purchased from other sectors. Then

$$y_s(i) = z_{Ls}(i)\left[\frac{\ell_s(i)}{\alpha_s}\right]^{\alpha_s}\left[\frac{X_s(i)}{1-\alpha_s}\right]^{1-\alpha_s} \quad \text{or} \quad y_s(i) = z_{Ms}(i)\left[\frac{m_s(i)}{\alpha_s}\right]^{\alpha_s}\left[\frac{X_s(i)}{1-\alpha_s}\right]^{1-\alpha_s}, \tag{B182}$$

$$X_s(i) = \prod_k \left[\frac{x_{ks}(i)}{\omega_{ks}}\right]^{\omega_{ks}}, \qquad \omega_{ks} \geq 0, \qquad \sum_k \omega_{ks} = 1, \qquad Q_s \equiv \prod_k P_k^{\omega_{ks}}. \tag{B183}$$

The parameter $\alpha_s \in (0,1]$ is the share of a task's cost paid to the worker or to the AI rather than to suppliers; we call it the value added share. It and the bundle weights ω_{ks} are the same for the two techniques within a sector and may differ across sectors, and setting $\alpha_s = 1$ for every s returns the model of Section 4. Cost minimization gives the unit cost of one efficiency unit as $w^{\alpha_s} Q_s^{1-\alpha_s}$ for workers and $r^{\alpha_s} Q_s^{1-\alpha_s}$ for AI, where Q_s is the price of the bundle, so competition prices each task at

$$p_s(i) = Q_s^{1-\alpha_s} \min\left\{\frac{w^{\alpha_s}}{z_{Ls}(i)}, \frac{r^{\alpha_s}}{z_{Ms}(i)}\right\}. \tag{B184}$$

Cobb–Douglas cost shares imply that a task's revenue $p_s(i)y_s(i)$ pays the share α_s to the worker or the AI that produces it and the share $(1-\alpha_s)\omega_{ks}$ to sector k. Productivity draws follow the joint law of Appendix A sector by sector, with scales T_{Ls} and T_{Ms} and a common ψ, so equation (A14) gives $\Pr[z_{Ls}(i)/z_{Ms}(i) \geq x] = T_{Ls}/(T_{Ls} + T_{Ms}x^{\psi})$ in every sector.

Weights. Let $\Gamma_{ks} \equiv (1-\alpha_s)\omega_{ks}$ be the share of sector s's gross output spent on good k, let Γ be the matrix with these entries, and note that $\alpha_s = 1 - \sum_k \Gamma_{ks}$. Because every α_s is positive, every column of Γ sums to less than one, so $(I-\Gamma)^{-1} = \sum_{n\geq 0}\Gamma^n$ exists and is nonnegative. Define

$$\lambda \equiv (I-\Gamma)^{-1}\beta, \qquad \Lambda \equiv \sum_s \lambda_s. \tag{B185}$$

Here β denotes the final demand shares in equation (C4). We use three facts below. First, $\sum_s \alpha_s\lambda_s = 1$: column sums give $\mathbf{1}'(I-\Gamma) = \alpha'$, so $\alpha'\lambda = \mathbf{1}'(I-\Gamma)\lambda = \mathbf{1}'\beta = 1$. Second, $\lambda_s \geq \beta_s$ for every s, because $\lambda = \beta + \Gamma\lambda$ and $\Gamma\lambda \geq 0$. Third, $\Lambda = 1 + \mathbf{1}'\Gamma\lambda \geq 1$, with equality if and only if $\Gamma = 0$, because $\lambda \geq \beta > 0$ makes $\Gamma\lambda = 0$ equivalent to $\Gamma = 0$.

Assignment. By equation (B184), workers produce task i of sector s if and only if $w^{\alpha_s}/z_{Ls}(i) \leq r^{\alpha_s}/z_{Ms}(i)$; the bundle price cancels because both techniques use the same bundle. The cutoff is therefore $z_{Ls}(i)/z_{Ms}(i) \geq (w/r)^{\alpha_s}$, and equation (A14) at $x = (w/r)^{\alpha_s}$ gives equation (28). Differentiating,

$$\frac{\partial \ln \pi_{Ms}}{\partial \ln(w/r)} = \alpha_s \psi \, \pi_{Ls}, \qquad \frac{\partial \ln \pi_{Ms}}{\partial \ln T_{Ms}} = \pi_{Ls}. \tag{B186}$$

To see why the first elasticity is $\alpha_s \psi$ rather than ψ, note that only the fraction α_s of a task's cost goes to the worker or to the AI. A one percent rise in w/r therefore raises the cost of the worker technique relative to the AI technique by only α_s percent, and tasks reassign accordingly. The response to T_{Ms} is unchanged.

Sector price index. By equation (B184), $p_s(i)/Q_s^{1-\alpha_s} = \min\{w^{\alpha_s}/z_{Ls}(i), r^{\alpha_s}/z_{Ms}(i)\}$ is the winning price of Appendix A, that is, the price of the cheaper technique, with the factor prices (w, r) replaced by $(w^{\alpha_s}, r^{\alpha_s})$ and the scales (T_L, T_M) by (T_{Ls}, T_{Ms}). The results of that appendix depend on factor prices only through the competitiveness terms of equation (A16), so they apply with $A_L = T_{Ls} w^{-\alpha_s \psi}$, $A_M = T_{Ms} r^{-\alpha_s \psi}$, and

$$A = \Phi_s \equiv T_{Ls} w^{-\alpha_s \psi} + T_{Ms} r^{-\alpha_s \psi}. \tag{B187}$$

Equation (A20) gives $\pi_{Ls} = T_{Ls} w^{-\alpha_s \psi}/\Phi_s$, which is equation (28) again, and equations (A17) and (A18) give

$$\Pr[p_s(i) > p] = \exp\left[-\Phi_s^{1-\rho}\left(\frac{p}{Q_s^{1-\alpha_s}}\right)^{\theta}\right], \qquad \frac{P_s}{Q_s^{1-\alpha_s}} = \gamma_s \Phi_s^{-1/\psi},$$

with γ_s the constant of equation (A22) evaluated at σ_s. Hence

$$P_s = \gamma_s Q_s^{1-\alpha_s} \Phi_s^{-1/\psi} = \gamma_s \, w^{\alpha_s} Q_s^{1-\alpha_s} \left(\frac{\pi_{Ls}}{T_{Ls}}\right)^{1/\psi}. \tag{B188}$$

Equation (B188) states that the sector price is the unit cost of a task at productivity one, $w^{\alpha_s} Q_s^{1-\alpha_s}$, times a selection term that falls as the share of tasks done by workers falls. Equation (A19) shows that the distribution of the winning price conditional on the producing technique is the same for workers and for AI, so tasks done by workers account for the share π_{Ls} of sector revenue $P_s Y_s$.

Payments and market clearing. Let $R_s \equiv P_s Y_s$ denote sector revenue. Zero profits and the Cobb–Douglas cost shares hold task by task, and aggregating over the tasks of sector s with the revenue shares just derived gives

$$w L_s = \alpha_s \pi_{Ls} R_s, \qquad r M_s = \alpha_s \pi_{Ms} R_s, \qquad P_k X_{ks} = \Gamma_{ks} R_s. \tag{B189}$$

Market clearing for good k reads $R_k = \beta_k Y + \sum_s \Gamma_{ks} R_s$, that is, $(I - \Gamma) R = \beta Y$, so

$$R_s = \lambda_s Y, \qquad Y = \sum_s \alpha_s R_s = wL + rM, \tag{B190}$$

where the last equality uses $\alpha'\lambda = 1$. Equation (B190) states that sector s sells λ_s dollars of gross output per dollar of GDP, so λ_s is the sector's Domar weight and Λ is the ratio of gross output to GDP; λ_s exceeds β_s because the sector sells to other sectors as well as to final users. Summing the payments to workers and to AI, $wL = \sum_s \alpha_s\lambda_s\pi_{Ls}Y$ and $rM = \sum_s \alpha_s\lambda_s\pi_{Ms}Y$. Labor's share of GDP is therefore the value added weighted share of tasks done by workers, $s_L = \sum_s \alpha_s\lambda_s\pi_{Ls}$, which extends Corollary 1. Dividing rM by $wL + rM$ gives factor market clearing,

$$\sum_s \alpha_s\lambda_s\pi_{Ms} = \frac{M}{M + (w/r)L}, \tag{B191}$$

which is equation (24) with $\alpha_s\lambda_s$ in place of β_s and π_{Ms} from equation (28). Its left side increases strictly from 0 to 1 as w/r rises from 0 to ∞, and its right side decreases strictly from 1 to 0, so equation (B191) has a unique solution $w/r > 0$. Dividing $wL_s = \alpha_s\lambda_s\pi_{Ls}Y$ by wL gives the allocation of workers across sectors,

$$\frac{L_s}{L} = \frac{\alpha_s\lambda_s\pi_{Ls}}{\sum_k \alpha_k\lambda_k\pi_{Lk}}, \qquad \frac{M_s}{M} = \frac{\alpha_s\lambda_s\pi_{Ms}}{\sum_k \alpha_k\lambda_k\pi_{Mk}}, \tag{B192}$$

which is equation (B74) with $\alpha_s\lambda_s$ in place of β_s.

We can now state the real wage. With intermediate inputs, the price of the final good depends on the wage through each sector's own costs and through the costs of the sectors it buys from, and the following proposition shows that these channels combine into a single weighted product.

Proposition 1 (Real wage and output). *In equilibrium the real wage is given by equation (29), equivalently*

$$w = \prod_s \left\{ \gamma_s^{-1/\alpha_s} \left(\frac{T_{Ls}}{\pi_{Ls}} \right)^{1/(\alpha_s\psi)} \right\}^{\alpha_s\lambda_s}, \tag{B193}$$

and real output is $Y = w\,[L + M/(w/r)]$.

Proof. Taking logs of equation (B188) and using $Q_s = \prod_k P_k^{\omega_{ks}}$,

$$\ln P_s = \ln \gamma_s + \alpha_s \ln w + \sum_k \Gamma_{ks} \ln P_k + \frac{1}{\psi} \ln \frac{\pi_{Ls}}{T_{Ls}}. \tag{B194}$$

Let $\mathbf{p}$ be the vector with entries $\ln P_s$ and $\mathbf{c}$ the vector with entries $c_s \equiv \ln \gamma_s + \alpha_s \ln w + \psi^{-1} \ln(\pi_{Ls}/T_{Ls})$. The system reads $\mathbf{p} = \mathbf{c} + \Gamma'\mathbf{p}$, so $\mathbf{p} = (I - \Gamma')^{-1}\mathbf{c}$ and $0 = \ln P = \beta'\mathbf{p} = [(I - \Gamma)^{-1}\beta]'\mathbf{c} = \lambda'\mathbf{c}$. Since $\lambda'\alpha = 1$, the wage enters the normalization with coefficient one:

$$0 = \ln w + \sum_s \lambda_s \left[\ln \gamma_s + \frac{1}{\psi} \ln \frac{\pi_{Ls}}{T_{Ls}} \right].$$

Rearranging gives equation (29), equivalently equation (B193). Real output is equation (B190). ◻

Proposition 1 states that the real wage is a weighted product of sector terms, each of which rises as the share of tasks done by workers in that sector falls, as in Section 4. Intermediate inputs change the weights. In the form of equation (29), sector s enters with the exponent λ_s/ψ rather than β_s/ψ, because a cost saving in sector s lowers the price of the final good directly and again through every sector that buys from s; in the form of equation (B193), the same weight is written as the value added weight $\alpha_s\lambda_s$ on a sector term with the exponent $1/(\alpha_s\psi)$.

Three consequences follow. First, if $\alpha_s = 1$ for every s, then $\Gamma = 0$, $\lambda = \beta$, and $T_{Ls}/\pi_{Ls} = T_{Ls} + T_{Ms}(w/r)^{\psi}$ by equation (28), so equations (28), (B191), and (29) reduce to equations (23), (24), and (25); with one sector, equation (29) is equation (11) with ψ replaced by $\alpha\psi$ and γ by $\gamma^{1/\alpha}$. Second, between two equilibria that differ in T_{Ms} or M but share T_{Ls}, α_s, ω_{ks}, β_s, and ψ, taking differences of logs in equation (29) gives equation (30); in the exact hat notation of Appendix C, $\hat{w} = \prod_s \hat{\pi}_{Ls}^{-\lambda_s/\psi}$. Third, equations (28), (B191), (B193), and the output identity are the system of Section 4 after replacing $(\beta_s, \psi, \gamma_s)$ by $(\alpha_s\lambda_s, \alpha_s\psi, \gamma_s^{1/\alpha_s})$, with the dispersion parameter now sector specific. When $\alpha_s = \alpha$ for every s, $\alpha = 1/\Lambda$ because $\sum_s \alpha_s\lambda_s = 1$, the effective dispersion ψ/Λ is common to all sectors, and Theorem 3 and Corollary 2 hold with ψ replaced by ψ/Λ and β_s by $\alpha_s\lambda_s$; equation (31) is equation (27) after this substitution. The rest of this section treats the case in which α_s differs across sectors.

Growth when AI improves at rates that differ across sectors. Let $T_{Ms,t} = T_{Ms,0}A_t^{\nu_s}$ as in equation (26): a common AI productivity index A_t increases without bound under the regularity conditions of Appendix B.5, and the exponent $\nu_s \geq 0$, positive in at least one sector, measures how much of that progress reaches sector s. Write $x \equiv \mathrm{d}\ln(w/r)/\,\mathrm{d}\ln A$ for the response of the wage relative to the AI rental and $\bar{m} \equiv \sum_s \alpha_s\lambda_s\pi_{Ms} = M/[M + (w/r)L]$ for AI's share of GDP. At every date,

$$\frac{\mathrm{d}\pi_{Ms}}{\mathrm{d}\ln A} = \pi_{Ms}\pi_{Ls}\left[\nu_s + \alpha_s\psi\, x\right], \tag{B195}$$

$$x = -\frac{\sum_s \alpha_s\lambda_s\pi_{Ms}\pi_{Ls}\nu_s}{\bar{m}(1-\bar{m}) + \psi\sum_s \alpha_s^2\lambda_s\pi_{Ms}\pi_{Ls}}, \tag{B196}$$

$$\frac{\mathrm{d}\ln w}{\mathrm{d}\ln A} = \frac{1}{\psi}\sum_s \lambda_s\pi_{Ms}\nu_s + \bar{m}\, x, \tag{B197}$$

$$\frac{\mathrm{d}\ln Y}{\mathrm{d}\ln A} = \frac{1}{\psi}\sum_s \lambda_s\pi_{Ms}\nu_s. \tag{B198}$$

Equation (B195) is the log derivative of equation (28). To obtain equation (B196), differentiate equation (B191) with respect to $\ln A$: the right side has derivative $-\bar{m}(1-\bar{m})x$, and substituting equation (B195) on the left and solving for x gives the result. To obtain equation (B197), note that equations (29) and (B195) give

$$\mathrm{d}\ln w = -\frac{1}{\psi}\sum_s \lambda_s \,\mathrm{d}\ln \pi_{Ls}, \qquad \frac{\mathrm{d}\ln \pi_{Ls}}{\mathrm{d}\ln A} = -\pi_{Ms}\left[\nu_s + \alpha_s \psi\, x\right];$$

using equation (B191) yields the result. Since $\mathrm{d}\ln[L + M/(w/r)]/\,\mathrm{d}\ln(w/r) = -\bar{m}$, the terms in $\bar{m}x$ cancel in $\mathrm{d}\ln Y/\,\mathrm{d}\ln A$. Equation (B198) is Hulten's theorem for this economy: output growth is the Domar weighted sum of sector productivity growth at fixed factor prices, since $\psi^{-1}\pi_{Ms}\nu_s$ is the derivative of $\ln(T_{Ls}/\pi_{Ls})^{1/\psi}$ with respect to $\ln A$ holding w/r fixed. With $\alpha_s = 1$, equations (B195)–(B198) are the elasticity identities of Step 1 of the proof in Appendix B.5.

We can now state how real wages and output grow in the long run when the value added share differs across sectors. The result turns on the ratio $\nu_s/(1+\alpha_s\psi)$, which compares how fast AI improves in sector s with how elastically the sector's tasks reassign in response.

Proposition 2 (Limits with heterogeneous value added shares)**.** *Define* $\kappa_s \equiv \nu_s/(1+\alpha_s\psi)$, $\kappa^* \equiv \min_s \kappa_s$, *and* $\mathcal{S}^* \equiv \arg\min_s \kappa_s$.

(a) If $\nu_s > 0$ *for every* s, *then* $w/r \sim c^* A^{-\kappa^*}$ *for a constant* $c^* > 0$, $\pi_{Ms} \to 1$ *for every* s, *labor's share of GDP converges to zero,* $L_s/L \to 0$ *for every* $s \notin \mathcal{S}^*$, $x \to -\kappa^*$, *and*

$$\frac{g_w}{g_Y} \longrightarrow 1 - \frac{\psi\,\kappa^*}{\sum_s \lambda_s \nu_s} \in \left[\frac{\Lambda}{\Lambda+\psi}, 1\right). \tag{B199}$$

The lower endpoint is attained if and only if ν_s *is proportional to* $1+\alpha_s\psi$.

(b) If $\nu_s = 0$ *for some* s, *then* w/r *converges to a finite positive limit, labor's share of GDP converges to a positive limit,* $L_s/L \to 0$ *for every* s *with* $\nu_s > 0$, $x \to 0$, *and* $g_w/g_Y \to 1$.

Proposition 2 states that when AI improves in every sector, the wage falls without bound relative to the AI rental, workers end up in the sectors with the smallest κ_s, and real wages grow at a fraction of the rate of output that is at least $\Lambda/(\Lambda+\psi)$. When some sector sees no AI progress, that sector retains workers, the wage relative to the AI rental settles at a positive value, and real wages eventually grow as fast as output. With $\alpha_s = \alpha = 1/\Lambda$ for every s, $\kappa^* = \Lambda \min_s \nu_s/(\Lambda+\psi)$, so equation (B199) together with part (b) is equation (31). With $\alpha_s = 1$ for every s, $\Lambda = 1$ and the proposition is Theorem 3. In what follows, $f \sim g$ means $f/g \to 1$ and $f \asymp g$ means that f/g is bounded above and below by positive constants.

Proof. The proof proceeds in three steps. First, we find the rate at which the wage falls relative to the AI rental in part (a). Second, we derive the limiting task shares, employment, and growth ratio in part (a). Third, we prove part (b). All limits are taken as $A \to \infty$, and $o(1)$ denotes a term that converges to zero in that limit.

Step 1. The relative factor price in part (a). Write equation (B191) as

$$\sum_s \alpha_s \lambda_s \frac{T_{Ls}}{T_{Ls} + T_{Ms,0} A^{\nu_s} (w/r)^{\alpha_s \psi}} = \frac{(w/r)L}{M + (w/r)L}. \tag{B200}$$

Evaluate both sides at $w/r = cA^{-\kappa^*}$ with c restricted to a closed interval $[c_1, c_2]$, $0 < c_1 < c_2 < \infty$. Since

$$\nu_s - \kappa^* \alpha_s \psi = (1 + \alpha_s \psi)(\kappa_s - \kappa^*) + \kappa^* \geq \kappa^*,$$

with equality if and only if $s \in \mathcal{S}^*$, the term for sector s on the left of equation (B200) equals

$$\alpha_s \lambda_s \frac{T_{Ls}}{T_{Ms,0}} \, c^{-\alpha_s \psi} A^{-(\nu_s - \kappa^* \alpha_s \psi)} \, [1 + o(1)]$$

uniformly in c. The terms with $s \notin \mathcal{S}^*$ are $o(A^{-\kappa^*})$, and the right side equals $cA^{-\kappa^*} L/M \, [1 + o(1)]$. Dividing by $A^{-\kappa^*}$, the limiting equation is

$$\sum_{s \in \mathcal{S}^*} \alpha_s \lambda_s \frac{T_{Ls}}{T_{Ms,0}} c^{-\alpha_s \psi} = c \, \frac{L}{M}. \tag{B201}$$

Its left side decreases and its right side increases in c, so it has a unique root $c^* > 0$. Fix $\varepsilon \in (0, 1)$. For all sufficiently large A, the left side of equation (B200) exceeds the right side at $w/r = (1 - \varepsilon)c^* A^{-\kappa^*}$ and falls short of it at $w/r = (1 + \varepsilon)c^* A^{-\kappa^*}$. Because the left side decreases and the right side increases in w/r, the equilibrium lies between these two values. Letting $\varepsilon \downarrow 0$ gives $w/r \sim c^* A^{-\kappa^*}$.

Step 2. Shares, allocation, and the limit ratio in part (a). Consequently

$$\pi_{Ls} \asymp A^{-(\nu_s - \kappa^* \alpha_s \psi)} \longrightarrow 0 \quad \text{for every } s,$$

so $\bar{m} \to 1$ and labor's share of GDP, $(w/r)L/[M + (w/r)L]$, converges to zero. By equation (B192), $L_s/L \to 0$ for $s \notin \mathcal{S}^*$, because π_{Ls} then decays faster than $A^{-\kappa^*}$ while $\sum_k \alpha_k \lambda_k \pi_{Lk}$ is at least of order $A^{-\kappa^*}$. In equation (B196), $\bar{m}(1 - \bar{m}) = \sum_s \alpha_s \lambda_s \pi_{Ls}[1 + o(1)]$ and $\pi_{Ms}\pi_{Ls} = \pi_{Ls}[1 + o(1)]$; terms with $s \notin \mathcal{S}^*$ are negligible relative to terms with $s \in \mathcal{S}^*$, and $\nu_s = \kappa^*(1 + \alpha_s \psi)$ on $\mathcal{S}^*$. Hence

$$x = -\kappa^* \frac{\sum_{s \in \mathcal{S}^*} \alpha_s \lambda_s \pi_{Ls}(1 + \alpha_s \psi)}{\sum_{s \in \mathcal{S}^*} \alpha_s \lambda_s \pi_{Ls}(1 + \alpha_s \psi)} \, [1 + o(1)] \longrightarrow -\kappa^*.$$

Since $\pi_{Ms} \to 1$, equations (B197) and (B198) give

$$\frac{\mathrm{d} \ln Y}{\mathrm{d} \ln A} \longrightarrow \frac{1}{\psi} \sum_s \lambda_s \nu_s, \qquad \frac{\mathrm{d} \ln w}{\mathrm{d} \ln A} \longrightarrow \frac{1}{\psi} \sum_s \lambda_s \nu_s - \kappa^*.$$

Growth rates in calendar time are these derivatives multiplied by the growth rate of A, which cancels from the ratio. This proves equation (B199); the ratio is below one because $\kappa^* > 0$. For the lower bound, $\nu_k \geq \kappa^*(1 + \alpha_k\psi)$ for every k; multiplying by λ_k and summing, with $\sum_k \alpha_k\lambda_k = 1$,

$$\sum_k \lambda_k \nu_k \geq \kappa^* \left(\Lambda + \psi\right), \qquad \text{so} \qquad \frac{\psi\kappa^*}{\sum_k \lambda_k \nu_k} \leq \frac{\psi}{\Lambda + \psi},$$

with equality if and only if $\nu_k = \kappa^*(1 + \alpha_k\psi)$ for every k.

Step 3. Part (b). Let $\mathcal{K} \equiv \{s : \nu_s > 0\}$ and $\mathcal{N} \equiv \{s : \nu_s = 0\}$, both nonempty. For $s \in \mathcal{K}$, $\pi_{Ms} \to 1$ uniformly over w/r in any closed bounded interval within $(0, \infty)$; for $s \in \mathcal{N}$, π_{Ms} does not depend on A. The limiting clearing condition

$$\sum_{s\in\mathcal{K}} \alpha_s\lambda_s + \sum_{s\in\mathcal{N}} \alpha_s\lambda_s \frac{T_{Ms,0}(w/r)^{\alpha_s\psi}}{T_{Ls} + T_{Ms,0}(w/r)^{\alpha_s\psi}} = \frac{M}{M + (w/r)L}$$

has a unique root $(w/r)_\infty \in (0, \infty)$: its left side increases from $\sum_{s\in\mathcal{K}} \alpha_s\lambda_s < 1$ to 1 and its right side decreases from 1 to 0. Monotonicity of both sides of equation (B191) then places the equilibrium w/r within any fixed neighborhood of $(w/r)_\infty$ for all sufficiently large A, as in the proof of Theorem 3. Labor's share of GDP converges to $(w/r)_\infty L/[M + (w/r)_\infty L] > 0$. For $s \in \mathcal{K}$, $\pi_{Ls} \to 0$ while the denominator of equation (B192) converges to the positive limiting labor share, so $L_s/L \to 0$. In equation (B196) the numerator converges to zero, since $\nu_s = 0$ on $\mathcal{N}$ and $\pi_{Ls} \to 0$ on $\mathcal{K}$, while the denominator is bounded below by $\bar{m}(1 - \bar{m})$, which converges to a positive limit. Hence $x \to 0$, and equations (B197) and (B198) give $\mathrm{d}\ln w/\mathrm{d}\ln A \to \psi^{-1}\sum_{s\in\mathcal{K}} \lambda_s\nu_s = \lim \mathrm{d}\ln Y/\mathrm{d}\ln A$, so the ratio converges to one. □

Remark (Worst case and the sectors that keep workers). Without intermediate inputs, the worst case for real wage growth relative to output growth is equal progress across sectors (Corollary 2). When the value added share differs across sectors, the worst case is instead $\nu_s \propto 1 + \alpha_s\psi$: AI must improve faster in the sectors whose tasks reassign more elastically. Intermediate inputs raise the worst case ratio from $1/(1+\psi)$ to $\Lambda/(\Lambda+\psi)$. Employment concentrates in the sectors that minimize $\nu_s/(1 + \alpha_s\psi)$, and these need not be the sectors with the slowest AI progress, because a high value added share speeds reassignment, so a smaller wage decline absorbs a given ν_s and the sector retains its workers.

Remark (Draws that augment the primary input). Section 4.1 notes that the result depends on where the bundle of purchased inputs enters. Suppose the productivity draw augments only the primary input, the worker or the AI, so that the unit costs are $(w/z_{Ls})^{\alpha_s}Q_s^{1-\alpha_s}$ and $(r/z_{Ms})^{\alpha_s}Q_s^{1-\alpha_s}$. The cutoff is then $z_{Ls}/z_{Ms} \geq w/r$ and the share of tasks done by AI is equation (23). The winning price law of equation (A17) applies to $[p_s(i)/Q_s^{1-\alpha_s}]^{1/\alpha_s}$, and the price moment is finite when $\alpha_s(\sigma_s - 1) < \theta$, giving $P_s = \tilde{\gamma}_s Q_s^{1-\alpha_s}[w(\pi_{Ls}/T_{Ls})^{1/\psi}]^{\alpha_s}$ with a constant $\tilde{\gamma}_s$. The price system is again linear in logs, and the argument of Proposition 1 gives

$$w = \prod_s \left\{ \tilde{\gamma}_s^{-1/\alpha_s} \left(\frac{T_{Ls}}{\pi_{Ls}} \right)^{1/\psi} \right\}^{\alpha_s \lambda_s},$$

which is equation (C7) with c_s the unit cost of sector s's primary input composite. Under this reading, the factor-clearing system takes the same form as in the model without intermediate inputs, with its sector weights β_s replaced by $\alpha_s\lambda_s$, and Theorem 3 holds with $\alpha_s\lambda_s$ in place of β_s.

Remark (Different input bundles for workers and AI). Suppose the worker technique has shares $(\alpha_{Ls}, \omega^L_{ks})$ and the AI technique has shares $(\alpha_{Ms}, \omega^M_{ks})$, and let $a_{Ls} \equiv w^{\alpha_{Ls}} \prod_k P_k^{(1-\alpha_{Ls})\omega^L_{ks}}$ and $a_{Ms} \equiv r^{\alpha_{Ms}} \prod_k P_k^{(1-\alpha_{Ms})\omega^M_{ks}}$ denote the two unit costs at productivity one. The derivations above go through with $\Phi_s = T_{Ls}a_{Ls}^{-\psi} + T_{Ms}a_{Ms}^{-\psi}$, giving $\pi_{Ls} = T_{Ls}a_{Ls}^{-\psi}/\Phi_s$ and $P_s = \gamma_s a_{Ls}(\pi_{Ls}/T_{Ls})^{1/\psi}$, and the proof of Proposition 1 gives equation (29) with $\lambda^L \equiv (I - \Gamma^L)^{-1}\beta$ in place of λ, where $\Gamma^L_{ks} \equiv (1-\alpha_{Ls})\omega^L_{ks}$. As a function of the sector task shares, the real wage therefore depends only on the worker technique's input weights. The common bundle matters elsewhere. Without it, π_{Ms} depends on sector prices through a_{Ls}/a_{Ms} and not only on w/r, sector cost shares become averages of the two techniques' shares weighted by the task shares, and equation (B191) and the constant Domar weights in Proposition 2 no longer apply.

B.10 Trade Displacement with Blocks of Tasks

Environment. The economy is the multisector economy of Section 4 in the specific factors limit $\kappa \downarrow 1$ of Section 5, with home labor as the sector-specific factor. Home has ℓ_s efficiency units of labor attached to sector s; they earn the wage w_s and have productivity T_{Ls}. Foreign has L^* units of labor, paid the wage w^*, with productivity T^*_{Ls} in sector s. Foreign productivity grows with A: $T^*_{Ls,t} = T^*_{Ls,0}A_t^{\nu_s}$, with $\nu_s \geq 0$ and $\bar{\nu} = \sum_s \beta_s \nu_s > 0$. Within sector s, tasks are grouped in blocks of $k_s \geq 1$ tasks. Each task draws one productivity for home and one for foreign from the joint distribution of Section 2, which gives each country's task productivity a Fréchet distribution with shape parameter θ and allows the two draws to be correlated; draws are independent across tasks. Within a block, tasks are combined with a Cobb–Douglas aggregator with equal weights; blocks are combined with the CES aggregator of Section 2, with elasticity σ; and each block is produced by the country that produces it at lower cost. Final demand is Cobb–Douglas with weights β_s. As in Section 4, a single household receives all income, $PY = \sum_s w_s\ell_s + w^*L^*$, so Y is world output, and growth rates are $g_x = \dot{x}/x$. With $k_s = 1$ the economy is that of Section 5 at $\kappa \downarrow 1$, with $w^* = r$, $T^*_{Ls} = T_{Ms}$, and the foreign share $\pi^*_{Ls} = 1 - \pi_{Ls}$ in the role of π_{Ms}. We first show that the home share of sector spending, π_{Ls}, depends on wages and technology only through the sector's terms of trade, and that its elasticity with respect to the terms of trade converges to $k_s\psi$ as home's share of the sector goes to zero. Growth accounting then gives (34).

Productivity draws. Fix a sector and drop its index. Each task has a productivity at home, z_L, and a productivity abroad, z_L^*, drawn from the joint distribution of Section 2. As in Appendix A, we represent the pair as

$$z_L = \left(\frac{T_L\, S}{\xi}\right)^{1/\psi}, \qquad z_L^* = \left(\frac{T_L^*\, S}{\xi^*}\right)^{1/\psi}, \tag{B202}$$

where S, ξ and ξ^* are independent. The draws ξ and ξ^* are exponential with mean one and specific to a country. The draw S is common to the two countries at the same task and generates the correlation between home and foreign productivity: it follows the positive stable distribution with index $\theta/\psi = 1-\rho$ and Laplace transform (A2), and when $\theta = \psi$ we set $S \equiv 1$, in which case the two productivities are independent. Conditional on S, $\Pr[z_L \le x,\ z_L^* \le y \mid S] = \exp[-S(T_L x^{-\psi} + T_L^* y^{-\psi})]$, and integrating over S recovers the joint distribution of Section 2, $\exp[-(T_L x^{-\psi} + T_L^* y^{-\psi})^{\theta/\psi}]$.

Block costs. Let k denote the block size and (S_j, ξ_j, ξ_j^*) the draws at task j of a block. Since tasks enter the block with equal Cobb–Douglas weights, the cost of the block is the geometric mean of the task costs, w/z_{Lj} at home and w^*/z_{Lj}^* abroad. Substituting (B202),

$$c = w T_L^{-1/\psi} \prod_j \left(\frac{\xi_j}{S_j}\right)^{1/(k\psi)}, \qquad c^* = w^* T_L^{*-1/\psi} \prod_j \left(\frac{\xi_j^*}{S_j}\right)^{1/(k\psi)}.$$

The common draws S_j cancel from the cost ratio,

$$\frac{c}{c^*} = \varrho\, e^{-\Xi/(k\psi)}, \qquad \varrho = \frac{w}{w^*}\left(\frac{T_L^*}{T_L}\right)^{1/\psi}, \qquad \Xi = \sum_{j=1}^{k} \left(\ln \xi_j^* - \ln \xi_j\right), \tag{B203}$$

where ϱ is the sector's terms of trade, so that $\varrho^\psi = T_L^* w^{*-\psi}/(T_L w^{-\psi})$, and Ξ sums home's cost advantage over the block's tasks. Home supplies the block if and only if $\Xi \ge k\psi \ln \varrho$.

Sourcing probability. The difference between the logs of two independent exponential draws has the standard logistic distribution, so Ξ is the sum of k independent standard logistic variables. Let $F(x) = \Pr[\Xi \le x]$ denote its distribution function. The probability that home supplies a block is $1 - F(k\psi \ln \varrho)$. With $k = 1$, $1 - F(x) = 1/(1 + e^x)$ and the probability is $1/(1 + \varrho^\psi) = T_L w^{-\psi}/(T_L w^{-\psi} + T_L^* w^{*-\psi})$, the sourcing probability of Eaton and Kortum (2002) with trade elasticity ψ.

Spending share and sector price. Let $G(c, c^*)$ denote the joint distribution of the two block costs. With blocks aggregated by a CES function with elasticity σ, the sector price index is $P^{1-\sigma} = \int \min\{c, c^*\}^{1-\sigma}\, \mathrm{d}G(c, c^*)$. Splitting the integral by source,

$$P^{1-\sigma} = P_L^{1-\sigma} + P_L^{*1-\sigma}, \qquad P_L^{1-\sigma} = \int_{c \le c^*} c^{1-\sigma}\, \mathrm{d}G(c, c^*), \qquad P_L^{*1-\sigma} = \int_{c > c^*} c^{*1-\sigma}\, \mathrm{d}G(c, c^*), \tag{B204}$$

where P_L and P_L^* are the price indices of the blocks that home and foreign supply. Sector spending on a block is proportional to its cost raised to the power $1-\sigma$, so home's share of sector spending is $\pi_L = P_L^{1-\sigma}/P^{1-\sigma}$. In the Fréchet case $k = 1$, the distribution of a block's cost conditional on its source does not depend on the source (Eaton and Kortum, 2002), so the spending share equals the sourcing probability. With $k > 1$ this property fails: a block that home wins by a wide margin is also a cheap block. We therefore compute $P_L^{1-\sigma}$ directly. For a home block, $c^{1-\sigma} = (T_L w^{-\psi})^{-(1-\sigma)/\psi} \prod_j \xi_j^{(1-\sigma)/(k\psi)} S_j^{-(1-\sigma)/(k\psi)}$, and the condition $c \le c^*$ is $\Xi \ge k\psi \ln \varrho = k \ln[T_L^* w^{*-\psi}/(T_L w^{-\psi})]$, which involves only the ξ_j and ξ_j^*. The S_j therefore integrate out separately. Writing F_S for the distribution function of S,

$$\int_0^\infty s^{-(1-\sigma)/(k\psi)} \, \mathrm{d}F_S(s) = \frac{\Gamma\big(1+(1-\sigma)/(k\theta)\big)}{\Gamma\big(1+(1-\sigma)/(k\psi)\big)},$$

which follows from the Laplace transform (A2) by writing the power of s as a Gamma integral, $s^{-b} = \Gamma(b)^{-1}\int_0^\infty t^{b-1}e^{-ts}\,\mathrm{d}t$ for $b > 0$ and $s^b = b\,\Gamma(1-b)^{-1}\int_0^\infty t^{-b-1}(1-e^{-ts})\,\mathrm{d}t$ for $0 < b < 1$, integrating over s first, and substituting $u = t^{\theta/\psi}$. Integrating $\xi_j^{(1-\sigma)/(k\psi)}$ against the exponential density $e^{-\xi}$ gives $\Gamma(1+(1-\sigma)/(k\psi))$ times the Gamma density with shape $1+(1-\sigma)/(k\psi)$. Hence

$$P_L^{1-\sigma} = \zeta_k \left(T_L w^{-\psi}\right)^{-(1-\sigma)/\psi}\left[1 - F_\sigma\!\left(k \ln \frac{T_L^* w^{*-\psi}}{T_L w^{-\psi}}\right)\right], \qquad \zeta_k = \left[\Gamma\!\left(1+\frac{1-\sigma}{k\theta}\right)\right]^k, \tag{B205}$$

where F_σ is the distribution function of Ξ when each ξ_j is drawn from the Gamma distribution with shape $1+(1-\sigma)/(k\psi)$ rather than the exponential, so that $F_1 = F$. The constant ζ_k is finite if and only if $\sigma < 1+k\theta$, because $\int_0^\infty \xi^b e^{-\xi}\,\mathrm{d}\xi = \Gamma(1+b)$ requires $b > -1$ and the moment of order b of S is finite if and only if $b < \theta/\psi$; and the restriction $\sigma < 1+\theta$ of Section 2 guarantees $\sigma < 1+k\theta$. With $k = 1$, $\zeta_1 = \gamma^{1-\sigma}$, where γ is the price index constant (A22). The same computation with the roles of the two countries exchanged gives

$$P_L^{*1-\sigma} = \zeta_k \left(T_L^* w^{*-\psi}\right)^{-(1-\sigma)/\psi}\left[1 - F_\sigma\!\left(k \ln \frac{T_L w^{-\psi}}{T_L^* w^{*-\psi}}\right)\right]. \tag{B206}$$

Each component has the form of Eaton and Kortum (2002): the country's own term $Tw^{-\psi}$ raised to the power $-(1-\sigma)/\psi$, times the weighted probability that the country is the low cost supplier, which depends only on the ratio of the two terms. By (B204), the home share therefore depends on wages and technology only through the terms of trade,

$$\pi_L(\varrho) = \frac{\varrho^{1-\sigma}\left[1 - F_\sigma(k\psi \ln \varrho)\right]}{\varrho^{1-\sigma}\left[1 - F_\sigma(k\psi \ln \varrho)\right] + 1 - F_\sigma(-k\psi \ln \varrho)},$$

and is continuous with values in $(0, 1)$. With $k = 1$, $1 - F_\sigma(x) = (1 + e^x)^{-1-(1-\sigma)/\psi}$, so (B205) and (B206) give

$$P_L^{1-\sigma} = \zeta_1 \, T_L w^{-\psi} \big[T_L w^{-\psi} + T_L^* w^{*-\psi}\big]^{-1-(1-\sigma)/\psi}, \qquad P^{1-\sigma} = \zeta_1 \big[T_L w^{-\psi} + T_L^* w^{*-\psi}\big]^{-(1-\sigma)/\psi},$$

and hence

$$\pi_L = \frac{T_L w^{-\psi}}{T_L w^{-\psi} + T_L^* w^{*-\psi}}, \qquad P = \gamma \big[T_L w^{-\psi} + T_L^* w^{*-\psi}\big]^{-1/\psi}$$

for every σ: the share and price index of Eaton and Kortum (2002), and those of Section 2 with r and T_M in place of w^* and T_L^*. Shephard's lemma holds for any k. Differentiating $P^{1-\sigma}$ with respect to $\ln w$, the boundary between home and foreign blocks moves, but $c = c^*$ on the boundary, so only the factor $w^{1-\sigma}$ in the home costs contributes, $\partial P^{1-\sigma}/\partial \ln w = (1-\sigma)P_L^{1-\sigma}$; the same argument applies to T_L, w^* and T_L^*. Therefore

$$\frac{\partial \ln P}{\partial \ln w} = \pi_L, \qquad \frac{\partial \ln P}{\partial \ln T_L^*} = -\frac{1-\pi_L}{\psi}, \tag{B207}$$

and the elasticities with respect to w^* and T_L are $1 - \pi_L$ and $-\pi_L/\psi$.

Elasticity of the home share. Let f_σ denote the density of F_σ and $\varphi_\sigma(x) = f_\sigma(x)/[1 - F_\sigma(x)]$ its hazard rate, the rate at which the weighted probability that home supplies a block, $1 - F_\sigma(x)$, falls with the cost threshold. Raise w holding w^* and technology fixed, so that $\mathrm{d}\ln\varrho = \mathrm{d}\ln w$. Differentiating $\ln \pi_L = \ln P_L^{1-\sigma} - \ln P^{1-\sigma}$ with (B205) and (B207),

$$\varepsilon(\varrho) = -\frac{\mathrm{d}\ln \pi_L(\varrho)}{\mathrm{d}\ln \varrho} = k\psi\, \varphi_\sigma(k\psi \ln \varrho) - (1-\sigma)\big(1 - \pi_L(\varrho)\big), \tag{B208}$$

so that $1 + \varepsilon(\varrho) = \sigma(1 - \pi_L(\varrho)) + \pi_L(\varrho) + k\psi\, \varphi_\sigma(k\psi \ln \varrho) \geq \sigma(1 - \pi_L(\varrho)) + \pi_L(\varrho)$. With $k = 1$, $\varphi_\sigma(x) = [1 + (1-\sigma)/\psi]/(1 + e^{-x})$ and (B208) reduces to $\varepsilon(\varrho) = \psi(1 - \pi_L(\varrho))$. For general k, the limit of the elasticity as $\varrho \to \infty$ is determined by the limit of the hazard rate, which we now characterize. The hazard rate φ_σ is nondecreasing and converges to $1 + (1-\sigma)/(k\psi)$. When ξ_j has the Gamma distribution with shape $1 + (1-\sigma)/(k\psi)$ and ξ_j^* is exponential, $\ln \xi_j^*$ has density $e^{y - e^y}$ and $-\ln \xi_j$ has density $e^{-[1+(1-\sigma)/(k\psi)]y - e^{-y}}/\Gamma(1 + (1-\sigma)/(k\psi))$. Both densities are log-concave, so their sum Ξ has a log-concave density and a log-concave function $1 - F_\sigma$ (Prékopa, 1973), which is equivalent to a nondecreasing hazard rate. To find the limit φ_∞ of the hazard rate, note that a nondecreasing hazard rate makes the moment generating function $\int e^{tx}\,\mathrm{d}F_\sigma(x)$ finite for $0 \leq t < \varphi_\infty$ and infinite for $t \geq \varphi_\infty$. Since the draws are independent, the moment generating function is $[\Gamma(1+t)\Gamma(1 + (1-\sigma)/(k\psi) - t)/\Gamma(1 + (1-\sigma)/(k\psi))]^k$, which is finite if and only if $-1 < t < 1 + (1-\sigma)/(k\psi)$; hence $\varphi_\infty = 1 + (1-\sigma)/(k\psi)$. Next, by (B205) and (B206), the ratio $P_L^{*1-\sigma}/P_L^{1-\sigma} = \varrho^{\sigma-1}[1 - F_\sigma(-k\psi \ln \varrho)]/[1 - F_\sigma(k\psi \ln \varrho)]$ has elasticity $(\sigma - 1) + k\psi\, \varphi_\sigma(k\psi \ln \varrho) + k\psi\, f_\sigma(-k\psi \ln \varrho)/[1 - F_\sigma(-k\psi \ln \varrho)]$ with respect to ϱ, which converges to

$(\sigma-1)+k\psi\,\varphi_\infty = k\psi > 0$; so the ratio diverges, $\pi_L(\varrho)\to 0$ as $\varrho\to\infty$, and (B208) gives

$$\varepsilon(\varrho) \longrightarrow k\psi \qquad (\varrho\to\infty). \tag{B209}$$

As $\varrho\to 0$, two cases arise. If $\sigma\le 1$, then $\min\{c,c^*\}^{1-\sigma}\le c^{1-\sigma}$ gives $P^{1-\sigma}\le \zeta_k(T_L w^{-\psi})^{-(1-\sigma)/\psi}$ and hence $\pi_L(\varrho)\ge 1-F_\sigma(k\psi\ln\varrho)\to 1$. If $\sigma>1$, the ratio $P_L^{*1-\sigma}/P_L^{1-\sigma}$ is at most $\varrho^{\sigma-1}/[1-F_\sigma(k\psi\ln\varrho)]\to 0$; in both cases $\pi_L(\varrho)\to 1$.

Growth accounting. Normalize $w^*=1$ and let $\beta_+=\sum_{\nu_s>0}\beta_s>0$ be the spending weight of the sectors exposed to foreign progress. This normalization is a local unit for the derivation only; the results below are ratios of growth rates and do not depend on it. As in the proof of Theorem 4, strict convexity and the implicit function theorem make the equilibrium path differentiable in $\ln A$. Write $x'=\,\mathrm{d}\ln x/\,\mathrm{d}\ln A$, so that $g_x/g_A=x'$ along the path. Three conditions describe the equilibrium. Market clearing in sector s is $w_s\ell_s=\beta_s\pi_{Ls}(\varrho_s)PY$; income is $PY=\sum_s w_s\ell_s+L^*$; and sector prices are given by (B204), with the elasticities (B207), and $P=\prod_s P_s^{\beta_s}$. Write $\pi_s=\pi_{Ls}(\varrho_s)$ and $\varepsilon_s=\varepsilon_s(\varrho_s)$; by (B203), $(\ln\varrho_s)'=w_s'+\nu_s/\psi$.

Step 0: displacement. If $\nu_s>0$ then $\varrho_s\to\infty$, hence $\pi_s\to 0$ and $\varepsilon_s\to k_s\psi$ by (B209). To see this, suppose instead that ϱ_s stays below some $\bar\varrho$ along a sequence of dates. Then $\pi_s\ge\inf_{\varrho\le\bar\varrho}\pi_{Ls}(\varrho)>0$, since π_{Ls} is continuous and positive by (B204), so market clearing with $PY\ge L^*$ gives $w_s\ge\beta_s L^*\pi_s/\ell_s$, a lower bound on the home wage. But then (B203) gives $\ln\varrho_s\ge(\nu_s/\psi)\ln A-O(1)\to\infty$, a contradiction.

Step 1: clearing. Differentiating $\ln w_s=\ln PY+\ln\pi_{Ls}(\varrho_s)+\ln(\beta_s/\ell_s)$ with respect to $\ln A$ gives

$$(1+\varepsilon_s)\,w_s'=(PY)'-\varepsilon_s\nu_s/\psi.$$

Step 2: income. Differentiating the income identity gives $(PY)'=\sum_s\beta_s\pi_s w_s'$. Substituting Step 1,

$$(PY)'\Big[1-\sum_s\frac{\beta_s\pi_s}{1+\varepsilon_s}\Big]=-\frac{1}{\psi}\sum_s\frac{\beta_s\pi_s\varepsilon_s\nu_s}{1+\varepsilon_s}.$$

By (B208), $1+\varepsilon_s\ge\pi_s$, so each term in the bracket's sum is at most β_s. In the exposed sectors the terms converge to zero by Step 0, because $\pi_s\to 0$ while $1+\varepsilon_s\ge\min\{1,\sigma\}$. Hence the bracket is at least $1-\sum_{\nu_s=0}\beta_s-\sum_{\nu_s>0}\beta_s\pi_s/(1+\varepsilon_s)\to\beta_+>0$, while the right side converges to zero by Step 0. Therefore $(PY)'\to 0$, and Step 1 gives $w_s'\to -k_s\nu_s/(1+k_s\psi)$ for every s; in sectors with $\nu_s=0$ this uses $1+\varepsilon_s\ge\min\{1,\sigma\}$.

Step 3: prices and output. Equation (B207) gives $P_s'=\pi_s w_s'-(1-\pi_s)\nu_s/\psi\to-\nu_s/\psi$, so $P'\to-\bar\nu/\psi$ and $Y'=(PY)'-P'\to\bar\nu/\psi$, so $g_Y/g_A\to\bar\nu/\psi$ for every profile of block sizes.

Step 4: the share. $(w_s/P)'=w_s'-P'\to\bar\nu/\psi-k_s\nu_s/(1+k_s\psi)$. Dividing by Y',

$$\frac{g_{w_s}}{g_Y}=\frac{(w_s/P)'}{Y'}\longrightarrow 1-\frac{k_s\psi}{1+k_s\psi}\,\frac{\nu_s}{\bar\nu}.$$

This is (34). Since $\ln(w_s/P) = \int (w_s/P)' \,\mathrm{d}\ln A$ and $\ln A \to \infty$, the real wage rises without bound if $\nu_s/\bar{\nu} < 1 + 1/(k_s\psi)$ and falls to zero under the strict reverse inequality.

C Counterfactual Construction and Exact Hat Algebra

This appendix derives the counterfactual equilibrium with input output linkages, worker sorting, and fixed sector specific capital. We construct the baseline, specify AI technique changes, and solve for wages, capital rentals, prices, task assignments, and real value added. Installed AI services, sector capital stocks, human technology, and group populations remain fixed throughout.

C.1 Baseline Shares

There are S sectors and G worker groups. In addition to the input shares below, the counterfactual requires seven baseline objects. Three describe sectors: the final expenditure weight $\beta_s \geq 0$, with $\sum_s \beta_s = 1$; the AI task share π_{Ms}, with $\pi_{Ls} = 1 - \pi_{Ms}$; and the task bundle's share of primary cost $\theta_s \in (0, 1]$. The value added weights used below are implied by the expenditure and input shares: $v_s = \alpha_s \lambda_s$, where $\boldsymbol{\lambda} = (\mathrm{Id} - \Gamma)^{-1}\boldsymbol{\beta}$. The task shares divide payments to the worker AI bundle. Sector specific capital receives the remaining share $1 - \theta_s$ of primary payments.

The next two objects describe workers. The share of group g assigned to sector s is μ_{gs}, with $\sum_s \mu_{gs} = 1$. In the Roy sector choice model, μ_{gs} is also sector s's share of group g's wage bill. The share of the economy's total worker wage bill paid to group g is ω_g, with $\sum_g \omega_g = 1$.

The final two objects control how strongly assignments respond to prices. The parameter ψ governs the division of tasks between workers and AI, while κ_g governs how strongly workers in group g move across sectors when sector wages change.

All value added, task, group, and sorting shares are strictly positive, and $0 < \pi_{Ms} < 1$. We also require $\psi > 0$ and $\kappa_g > 1$. These conditions keep every sector and worker group active and ensure that average earnings are finite.

Let $I = Y$ denote aggregate factor cost income, where Y is real value added. Workers, installed AI services, and sector capital receive the baseline income shares

$$\begin{aligned} L_0 &\equiv \sum_s \theta_s v_s \pi_{Ls}, & M_0 &\equiv \sum_s \theta_s v_s \pi_{Ms}, \\ c_K &\equiv \sum_s (1-\theta_s) v_s, & L_0 + M_0 + c_K &= 1. \end{aligned} \tag{C1}$$

These shares are 54.1, 8.9, and 37 percent, respectively.

The baseline worker shares satisfy

$$L_0 \sum_g \omega_g \mu_{gs} = \theta_s v_s \pi_{Ls} \qquad \text{for every } s. \tag{C2}$$

The left side allocates the aggregate worker wage bill across groups and sectors. The right side is sector value added multiplied by the task bundle's primary cost share and workers' share within that bundle.

The same identity also implies each group's share of a sector's wage bill. That share is useful for reporting the baseline but is not an additional input to the counterfactual.

C.2 Consumption and Input Output Shares

We calibrate the consumption shares β_s and input output shares Γ_{ks} to the 2024 BEA accounts (U.S. Bureau of Economic Analysis, 2026). The sample covers the 71 published industries, with owner occupied housing and the 5 government industries. Government production follows the same system as production in other sectors. We use the 2024 matrix in *Use of Commodities by Industries, after Redefinitions, at Producers' Prices*. Factor cost value added is worker compensation plus gross operating surplus. Gross revenues X_s also include intermediate purchases. The shares are

$$\alpha_s = \frac{\text{value added}_s}{X_s}, \qquad \lambda_s = \frac{X_s}{\sum_k \text{value added}_k}. \tag{C3}$$

The matrix has commodities as rows and industries as columns; we match each commodity to the corresponding industry in the model.

We build Γ from the published commodity purchases. We exclude used goods and other inputs that have no corresponding industry. For each commodity, we multiply purchases by domestic production divided by total intermediate and final uses before deducting imports, capped at one. We then adjust rows and columns to match the model's input payments. Industry s's column payments are $(1-\alpha_s)\lambda_s$ as a share of aggregate factor cost value added. The row payments use the same factor across sectors, with a cap of λ_k on each row. We calibrate this factor to match aggregate column payments, then adjust each row and column by a factor until their payments match. The input share is $\Gamma_{ks} = \text{input payments}_{ks}/\lambda_s$.[1]

We calibrate β to match each industry's reported share of value added:

$$\begin{gathered} \alpha_s \left[(\mathrm{Id} - \Gamma)^{-1}\boldsymbol{\beta}\right]_s = \frac{\text{value added}_s}{\sum_k (\text{value added}_k)}, \\ \beta_s = \lambda_s - \sum_k \Gamma_{sk}\lambda_k. \end{gathered} \tag{C4}$$

The matrix Id has $\mathrm{Id}_{ks} = \mathbf{1}\{k = s\}$. The final consumption share is the industry's gross revenues less its output used by other industries, as a share of aggregate factor cost value added. The constructed shares are $\beta_s \geq 0$ and $\sum_s \beta_s = 1$; $\sum_s \lambda_s = 1.78$.

[1] Values that the data do not report are set to 0. Sectors at the upper limit are held fixed as we adjust the factor for the remaining sectors. There are 7 sectors at the upper limit.

Production combines the task technology of Section 4.1 with a separate sector capital bundle. The two bundles have cost shares θ_s and $1-\theta_s$ and use the same intermediate input shares. The capital bundle has unit cost $R_s^{\alpha_s}\prod_k P_k^{\Gamma_{ks}}$, where R_s is the rental on sector s's fixed capital stock K_s. Thus sector unit cost, up to a constant that cancels in changes, is

$$P_s = D_s R_s^{b_s}\prod_k P_k^{\Gamma_{ks}}\left[T_{Ls}w_s^{-\alpha_s\psi}+T_{Ms}r^{-\alpha_s\psi}\right]^{-\theta_s/\psi}, \quad a_s=\theta_s\alpha_s, \quad b_s=(1-\theta_s)\alpha_s. \tag{C5}$$

The constant $D_s>0$ collects time invariant aggregation terms. Capital is outside the comparison between worker and AI techniques, so

$$\pi_{Ls}=\frac{T_{Ls}w_s^{-\alpha_s\psi}}{T_{Ls}w_s^{-\alpha_s\psi}+T_{Ms}r^{-\alpha_s\psi}}, \qquad \pi_{Ms}=1-\pi_{Ls}. \tag{C6}$$

Define the normalized task cost $c_s=[T_{Ls}w_s^{-\alpha_s\psi}+T_{Ms}r^{-\alpha_s\psi}]^{-1/(\alpha_s\psi)}$. Then $P_s=D_sR_s^{b_s}c_s^{a_s}\prod_k P_k^{\Gamma_{ks}}$. Cobb Douglas demand keeps $X_s=\lambda_s I$ in every equilibrium. Since K_s is fixed and $R_sK_s=b_sX_s$, every active capital rental changes by $\hat{R}_s=\hat{I}$. Combining sector prices with the final expenditure weights gives

$$\begin{aligned}(\mathrm{Id}-\Gamma^\top)\log\widehat{\boldsymbol{P}} &= \boldsymbol{b}\log\hat{I}+\operatorname{diag}(\boldsymbol{a})\log\widehat{\boldsymbol{c}},\\ 0=\log\hat{P} &= \sum_s\lambda_s b_s\log\hat{I}+\sum_s\lambda_s a_s\log\hat{c}_s = c_K\log\hat{I}+\sum_s\theta_s v_s\log\hat{c}_s.\end{aligned} \tag{C7}$$

Sectors with $b_s=0$ make no capital payments and require no capital rental. The task and capital weights satisfy $\sum_s\theta_s v_s+c_K=1$.

The calibrated β and Γ match the reported value added shares and are fixed in the counterfactual equilibria.

C.3 Industry and Worker Data

Figure C1 shows how we combine the data across the 71 industries. The BEA accounts measure industry size and input shares; worker data measure employment, education, and wage bills; task data measure current AI use and exposure to future AI progress.[2]

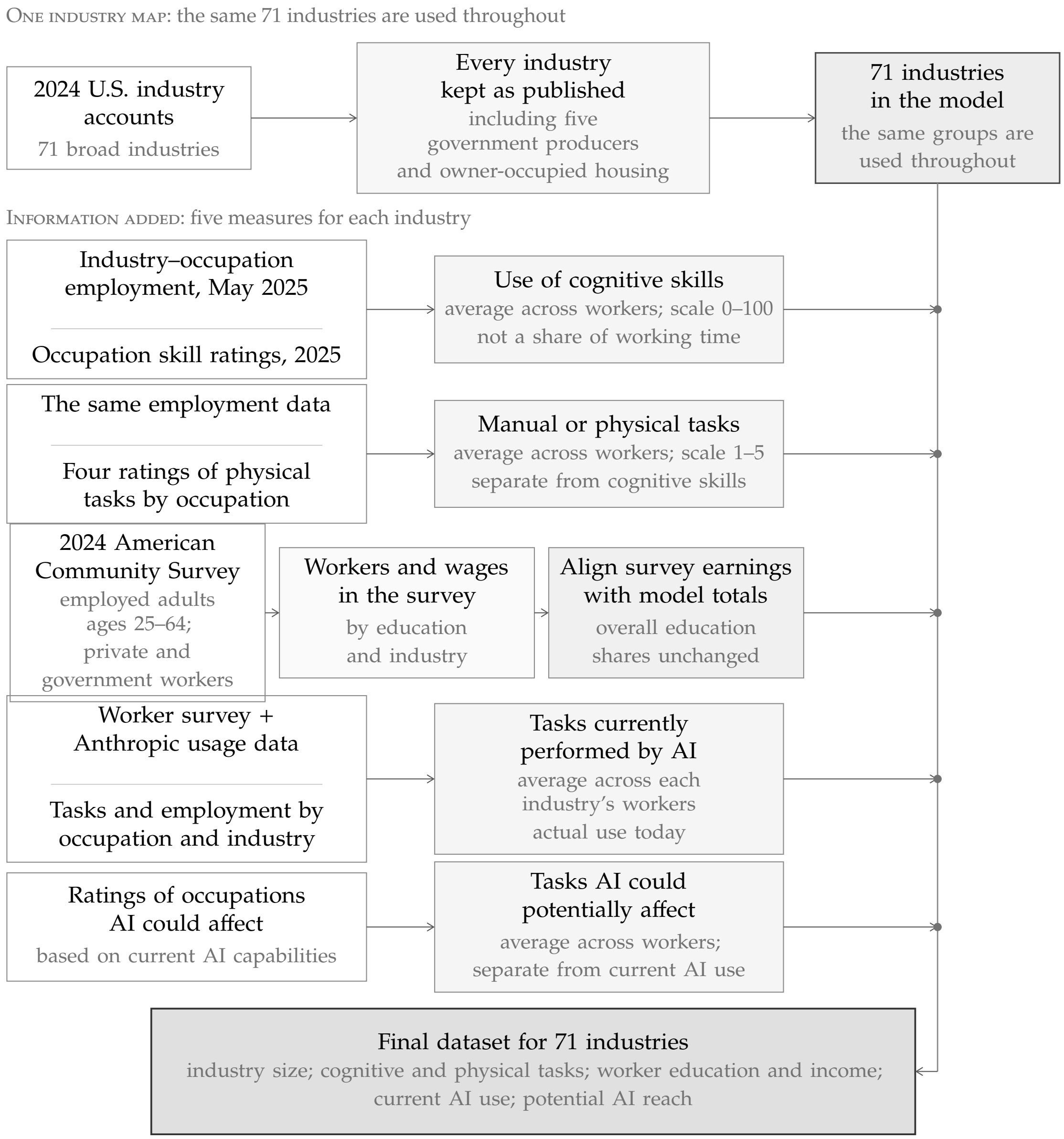
The industry map is the same for every measure. The skill indexes represent the tasks performed in each industry, and the education shares represent its workers. Current AI task shares set the

[2]To allow for capital income, we calibrate the share of primary non capital inputs as

$$\theta_s=\min\left\{\frac{w_s^DL_s^D}{\pi_{Ls}\alpha_sX_s},1\right\}, \qquad \pi_{Ls}=1-\pi_{Ms}. \tag{C8}$$

$w_s^DL_s^D$ is BEA worker compensation. Labor receives $\theta_s\alpha_s\pi_{Ls}X_s$, AI receives $\theta_s\alpha_s\pi_{Ms}X_s$, and sector specific capital receives $(1-\theta_s)\alpha_sX_s$. The share θ_s is set to one in 8 sectors, which leaves the aggregate model wage bill 1.29 percent below BEA compensation. We retain this accounting and match each industry's AI task share.

Figure C1. Industry Data Construction



Notes: The scores for cognitive and physical tasks use different units and are not a percentage of working time. An industry can score high on both. Education shares are based on the workers in the American Community Survey. We adjust survey earnings to match the model’s industry wage bills while keeping each education group’s overall income share. A second employment table covers industries or occupations not in the main employment data. Our calculation uses the BEA industry accounts; BLS employment, skills, and employment projection data; O*NET task descriptions; the 2024 American Community Survey; the July 2026 Epoch AI–Ipsos worker survey; the Anthropic Economic Index; and occupation AI capability ratings.

starting point for the simulations, while exposure sets where future AI progress is faster.

Skills and Exposure

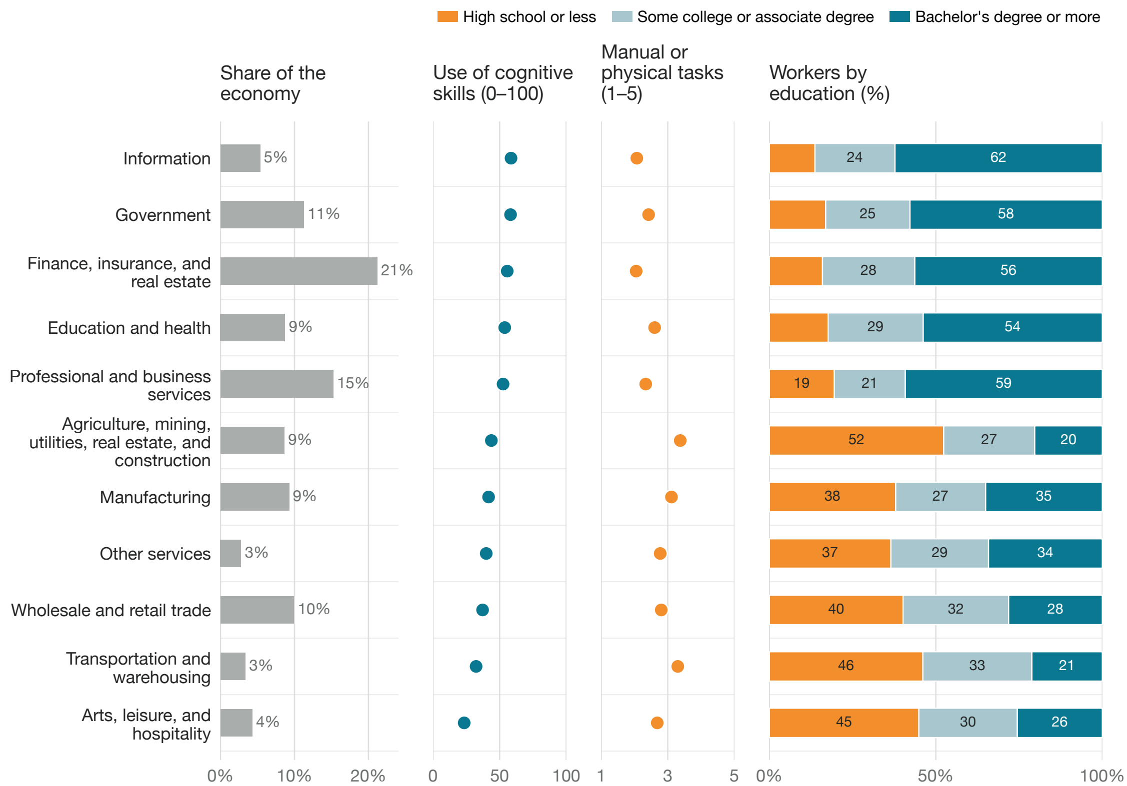
The cognitive score is the average of eight BLS skill scores in the 2025 Employment Projections data (U.S. Bureau of Labor Statistics, 2026*b*). The manual or physical score is the average importance score for physical tasks, handling objects, controlling machines, and operating vehicles and equipment in O*NET 31.0, published in August 2026 (National Center for O*NET Development, 2026). For both measures, we average detailed occupations within each BLS occupation. For missing occupation scores, we use averages across progressively broader occupations. We then average scores across occupations using industry employment, with only occupations represented in the scores. The cognitive score is on a 0 to 100 index and the manual or physical score is on a 1 to 5 index.[3]

Exposure measures how much AI could affect an occupation's tasks (Eloundou et al., 2024). We use human assessments of whether AI could halve a task's completion time, giving half weight to tasks requiring additional software. We average the scores across detailed occupations within each BLS occupation. For missing scores, we use averages across progressively broader occupations, then the US employment weighted average across occupations with scores. We average across occupations using the 2025 BLS Employment Projections industry employment shares for the 65 industries outside housing and government. For housing and government, we use ACS employment as detailed below. These exposure scores are separate from the current AI task shares.

Figure C2 combines the 71 industries into 11 rows, from the highest to the lowest cognitive score. The simulations use every industry as a separate sector. For each industry group in the figure, value added shares weight AI shares and exposure, employment shares weight the skill scores, and ACS employment shares represent worker education. The figure shows how industries of similar size can use different tasks and worker composition.

[3] U.S. Bureau of Labor Statistics (2026*b*) reports the 8 BLS skills. Industry employment is the May 2025 measure constructed below.

Figure C2. Industry Size, Tasks, and Worker Education



Notes: The figure reports, for each industry group, its share of value added, cognitive score, physical score, and worker education shares. Our calculation uses the 2024 BEA accounts, May 2025 BLS employment, 2025 BLS skill scores, O*NET 31.0, and the 2024 American Community Survey (U.S. Bureau of Economic Analysis, 2026; U.S. Bureau of Labor Statistics, 2026*c*,*b*; National Center for O*NET Development, 2026; U.S. Census Bureau, 2024).

Current AI Task Shares

The July 2026 Epoch AI Ipsos survey measures the starting AI shares for ten tasks (Epoch AI, 2026*b*). The survey covers 1,103 employed US adults. We use responses about tasks that respondents regularly perform, with 713 people and 2,272 responses entering estimation. We count a task as fully assigned to AI when respondents report that AI performs most or all of it, half assigned when AI performs part of it, and unassigned when they report no current AI use. Respondents without a reported occupation enter the survey's other occupations group.

The estimated AI share is $\pi^D_{M,bj}$ for task j and occupation group b. We estimate these shares using a weighted logit with separate occupation group and task parameters and a parameter for the full sample. The estimated share for the same task can vary by occupation group.[4]

[4] Survey weights have average one in each sample. We estimate the parameters to minimize $-\sum(\text{weight})[\text{score}\log\pi + (1-\text{score})\log(1-\pi)]$ across the included survey scores. We add $(\text{penalty}/2)(\text{parameter})^2$ for each occupation group and task parameter, but not for the parameter for the full sample. To set penalty for occupation groups and for tasks, we assign the sample at random to five groups, estimate on four groups, and test the fit on the remaining group. We

O*NET sets the task weight for each detailed occupation o (National Center for O*NET Development, 2026). Each task's weight is its relevance score divided by 100; tasks supplied by analysts without a relevance score receive weight one. When a detailed task belongs to several broader O*NET tasks, we divide its weight equally among them, including those outside the survey. Dividing the resulting weight by all task weights in occupation o gives the occupation's share of surveyed task j. We match the survey tasks to O*NET as follows:

Survey task	O*NET task
Maintaining operational records	Maintaining operational records
Developing organizational policies, systems, or processes	Developing organizational policies, systems, or processes
Reading documents or other materials	Reading documents or other materials
Determining values or prices of goods or services	Determining values or prices of goods or services
Analyzing data or evaluating data quality or accuracy	Evaluating data quality or accuracy
Designing computer or information systems or applications	Designing computer or information systems or applications
Preparing documentation for contracts, applications, or permits	Preparing documentation for contracts, applications, or permits
Explaining or providing information to clients or customers	Providing information to guests, clients, or customers
Advising others on business or operational matters	Advising others on business or operational matters
Assessing student capabilities, needs, or performance	Assessing student capabilities, needs, or performance

These ten tasks represent 10.25 percent of tasks using the employment weight below.

We combine the estimated survey shares using each occupation's task shares:

$$\pi_{Mo}^{10} = \frac{\sum_{j=1}^{10} (\text{task share}_{oj}) \pi_{M,b(o),j}^{D}}{\sum_{j=1}^{10} \text{task share}_{oj}}. \tag{C9}$$

The occupation group $b(o)$ is the survey group containing occupation o. Securities, commodities, and financial services sales agents enter the financial occupations group. For occupations with no

keep the two values with the best average fit across the five groups. The chosen values are 128 for occupation groups and 4 for tasks.

surveyed tasks, we use the same group's estimated shares, weighted by the matching task shares across the full O*NET sample.[5]

The Anthropic Economic Index lets us extend the measurement to the remaining tasks (Massenkoff and McCrory, 2026). Anthropic measures observed AI task use, giving greater weight to AI completing tasks than assisting with them. We match tasks across sources by their descriptions, first using identical descriptions and then close wording matches. The employment weighted average scores are 0.16 for surveyed tasks and 0.1 for the remaining tasks. We impose the same proportionality assumption in each occupation:

$$\pi_{Mo}^{\text{outside}} = \frac{\text{Anthropic average score outside the survey}}{\text{Anthropic average score in the survey}} \pi_{Mo}^{10} \simeq 0.66\, \pi_{Mo}^{10}. \tag{C10}$$

For each set of tasks, we use the weighted average Anthropic score, with the task weight divided by all task weights in occupation o, then multiplied by US employment in occupation o. Only tasks with an Anthropic score enter the averages; a reported 0 is included. The matching tasks account for 99.54 percent of weighted tasks.

The share for occupation o combines its surveyed and remaining tasks:

$$\pi_{Mo} = \sum_{j=1}^{10} (\text{task share}_{oj}) \pi_{M,b(o),j}^{D} + \left(1 - \sum_{j=1}^{10} \text{task share}_{oj}\right) \pi_{Mo}^{\text{outside}}. \tag{C11}$$

We average π_{Mo} across detailed occupations within each BLS occupation. For missing occupation matches, we use averages across progressively broader occupations within the survey occupation group. For remaining missing matches, we use the US employment weighted average across matched occupations. Six employment observations, accounting for 0.26 percent of employment, use this final average.

Industry employment is from May 2025 Occupational Employment and Wage Statistics (OEWS) (U.S. Bureau of Labor Statistics, 2026*c*). We combine detailed industries into the BEA sectors without counting employment twice. Education and hospital employment includes private employers; public employees enter the government sectors. We retain published occupation employment and allocate the gap between that employment and the industry's reported total using the 2025 BLS Employment Projections occupation shares (U.S. Bureau of Labor Statistics, 2026*b*). For agriculture, forestry, fishing, and private households outside OEWS coverage, we use employment from the BLS Employment Projections data. Housing and government, including the postal service, use the ACS occupation shares detailed below.

[5] For these occupations, the weight on surveyed task j is $\sum_o$ task share$_{oj}$ across O*NET occupations.

The constructed employment is L^D_{os} in occupation o and industry s. The sector AI share is

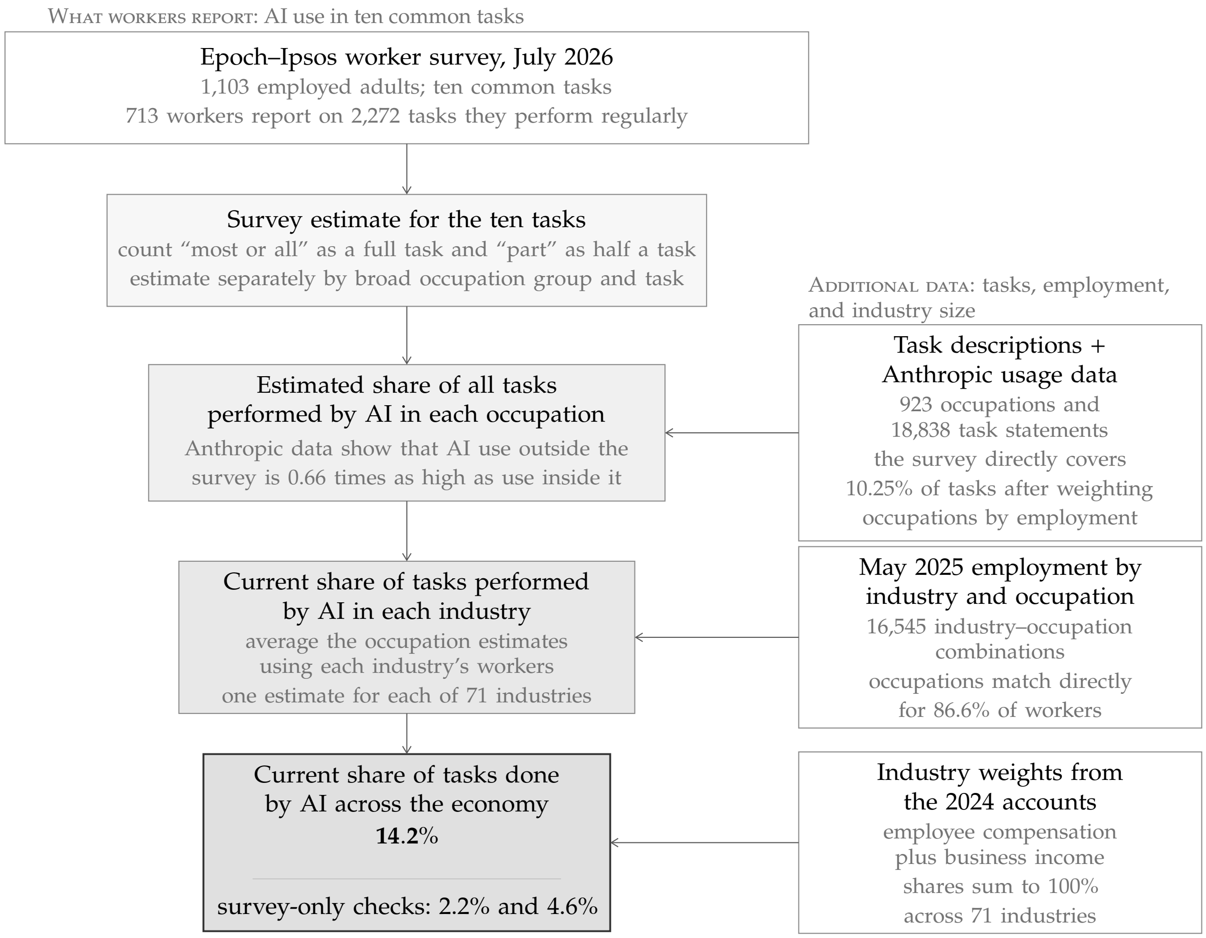
$$\pi_{Ms} = \sum_o \frac{L^D_{os}}{\sum_{o'} L^D_{o's}} \pi_{Mo}, \qquad \bar{\pi}_M = \sum_s \alpha_s \lambda_s \pi_{Ms}. \tag{C12}$$

Every industry has an AI task share above 0, and the industry size weighted average is 14.2 percent. Figure C3 shows the construction from surveyed tasks to sector shares. The survey sets AI use for the ten surveyed tasks, Anthropic data extend these shares to the remaining tasks, and occupation employment sets each industry's task mix.

With AI use set to 0 outside the surveyed tasks, the aggregate share is 2.2 percent. Counting only tasks that AI performs most or all of and applying the same AI rate to surveyed and remaining tasks in each occupation gives an aggregate share of 4.6 percent. The calibrated shares measure current use and set no ceiling on future AI use.[6]

[6]For this analysis, AI use on only some of a task receives a score of 0; the chosen values of penalty for occupation groups and tasks are ∞ and 8. For the range of estimated shares reported with Anthropic, we draw occupations into a sample 1,000 times. The estimated survey shares are fixed. Each draw includes the surveyed and remaining tasks of the drawn occupations, with the same employment weights, and produces sector AI shares. The reported range of 11.8 to 17.3 percent has 5 percent of the estimated shares below it and 5 percent above; the counterfactual equilibria use only the point estimate.

Figure C3. Current AI Task Shares



Notes: The survey measures AI use for ten tasks. Anthropic data extend the shares to the remaining tasks in each occupation. Employment shares combine the estimated occupation shares into industry shares, with an aggregate estimate of 14.2 percent. The survey only shares are 2.2 and 4.6 percent. Our calculation uses the Epoch AI Ipsos survey, Anthropic Economic Index, O*NET 31.0, May 2025 BLS employment, the 2025 BLS employment projection, and the 2024 BEA accounts (Epoch AI, 2026*b*; Massenkoff and McCrory, 2026; National Center for O*NET Development, 2026; U.S. Bureau of Labor Statistics, 2026*c*,*b*; U.S. Bureau of Economic Analysis, 2026).

Worker Shares

We build ω_g and μ_{gs} from the 2024 American Community Survey for the US (U.S. Census Bureau, 2024). The sample includes employed adults aged 25 to 64 with positive wage and salary income and reported industry and education. We group workers into high school or less, some college or an associate degree, and a bachelor's degree or more. Employment is the sum of survey weights. Wage bills sum annual wage and salary income after applying the Census income adjustment and multiplying by those weights. Government workers account for 16.9 percent of the wage bill.

We match each reported industry to the corresponding BEA sector. When a reported industry spans several BEA sectors, we divide its employment and wage bill using their shares of worker compensation. Unspecified manufacturing and retail employment is divided across the corresponding manufacturing and retail sectors. Combined textile and apparel industries, primary and fabricated metal industries, and securities and funds industries are divided in the same way. The reported real estate industry covers both other real estate and owner occupied housing, so their education mix comes from the same real estate workers.

Federal national security employees enter federal defense, and federal postal employees enter federal government enterprises. Other public administration employees enter general government at their level of government. We allocate remaining public employees according to government production in their reported industry in the 2024 BEA Make matrix, keeping federal employees separate from state and local employees. When no corresponding government production is reported, they enter general government.

Government occupation shares use the survey weights of government employees in each occupation. For the AI share and exposure of housing, we use property, real estate, and community association managers employed in real estate. Worker composition comes from the full real estate sample, as detailed above.

The wage bill of education group g in industry s is $(w_s L_{gs})^D$ in the data. The group's aggregate wage share and its industry wage shares are

$$\omega_g = \frac{\sum_s (w_s L_{gs})^D}{\sum_{h,s}(w_s L_{hs})^D}, \qquad \mu_{gs}^D = \frac{(w_s L_{gs})^D}{\sum_k (w_k L_{gk})^D}. \tag{C13}$$

The 3 group shares are 18.2, 21.4, and 60.4 percent, respectively.

We calibrate μ_{gs} by matching the model's industry wage bills and retain the group shares ω_g:

$$\begin{aligned} &\min_{\{\mu_{gs}>0\}} \sum_{g,s} \omega_g \mu_{gs} \log \frac{\mu_{gs}}{\mu_{gs}^D} \\ &\text{with } \sum_s \mu_{gs} = 1, \qquad \sum_g \omega_g \mu_{gs} = \frac{\theta_s \alpha_s \lambda_s \pi_{Ls}}{\sum_k \theta_k \alpha_k \lambda_k \pi_{Lk}}. \end{aligned} \tag{C14}$$

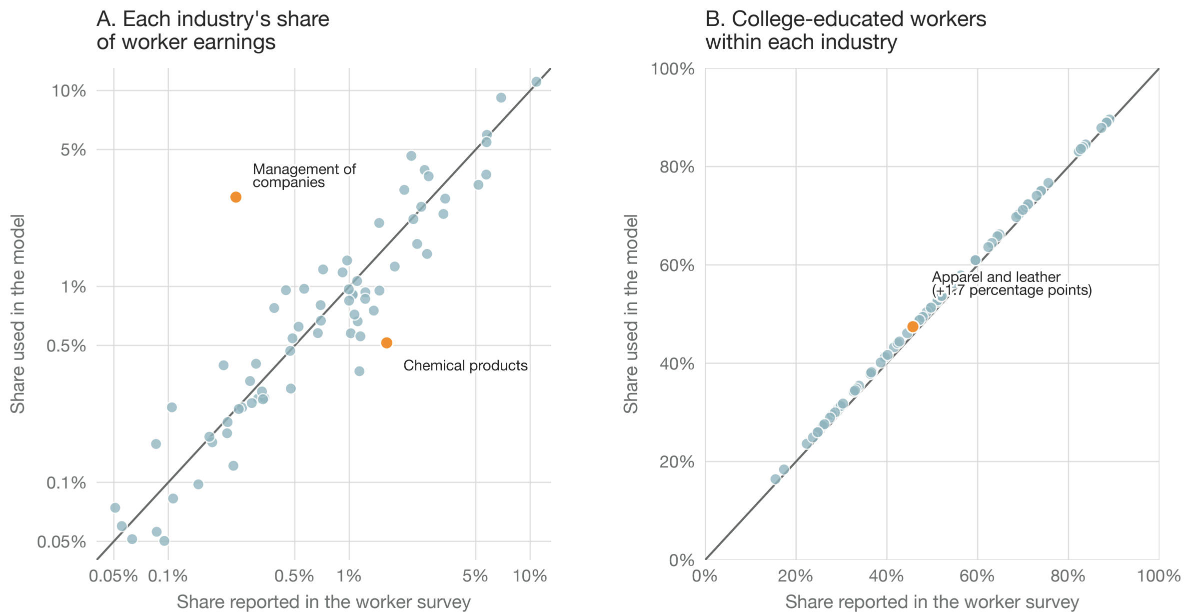
Starting from the reported group by industry wage shares, we adjust each group's wage bill to match its group share and each industry's wage bill to match its model share. μ_{gs} is consistent with (C2).

Figure C4 compares the reported and calibrated shares. Panel A shows industry wage bills as shares of aggregate worker payments; Panel B shows college educated workers' shares in industries. We shift 15.1 percent of aggregate worker payments across industries. Each education group's aggregate share is fixed.

The calibrated college educated shares are similar to their reported shares: the highest gap is 0.02. We match industry payments and retain similar worker composition in each industry.

We use $\psi = 1.36$ from Section 7 and set $\kappa_g = 1.5$ for every education group, following Galle, Rodríguez-Clare and Yi (2023).

Figure C4. Reported and Calibrated Worker Shares



Notes: Each circle is one industry. Panel A compares reported and calibrated industry wage shares, and Panel B compares the shares of workers with college degrees. Each line shows the same share in the data and model. The aggregate wage share of each education group is fixed while industry wage bills match the model. Our calculation uses the 2024 American Community Survey and the calibrated model (U.S. Census Bureau, 2024).

C.4 Tapering the AI Cost Path

Our same price regression covers model releases from November 20, 2024, through June 16, 2026. Over these 19 months, the estimated log cost decline is $g_0 \approx 1.57$, equivalent to 79.1 percent per year. A constant five year extrapolation would make a fixed task about 99.96 percent cheaper. The sample identifies the initial rate. It does not establish that this rate persists for five years.

Let c_t denote task cost, $p_{K,t}$ dollars per unit of compute, and q_t compute per task. The identity $c_t = p_{K,t} q_t$ implies that their log decline rates add. Our regression holds token prices fixed, so g_0 measures the decline in q_t. We use it as the initial rate of decline of the full task cost c_t and import g_H as the assumed long run floor rather than estimating it. AI chips purchased in 2023 to 2025 delivered about 49 percent more computing per dollar each year, corresponding to a 1.7 year doubling time (Somala, 2026). We use $g_H = \ln(2)/1.7 \approx 0.41$ for the durable hardware rate.[7] The compute required to reach a fixed language model capability has halved about every eight months (Ho et al., 2024). We collect the decline above the hardware rate into a fading excess that includes algorithmic and serving efficiency.

[7] Hobbhahn and Besiroglu (2022) estimate a 2.5 year doubling time for hardware price performance. For AI hardware, Del Sozzo et al. (2026) report 40.8 percent annual improvement in compute per dollar, or 49.7 percent when hardware does no work on zero values, which implies doubling times of 2.03 and 1.72 years. Some of their price sources are the same as those of Hobbhahn and Besiroglu (2022).

Learning curves motivate the taper. For any layer, the annual log cost decline equals its learning elasticity times the growth rate of its cumulative experience stock (Wright, 1936; Arrow, 1962; Nagy et al., 2013). The chip production stock is mature and advances on a comparatively steady clock. The AI experience stock is young. Its doublings lengthen when adoption and investment growth slow. A constant decline is appropriate only while cumulative use grows exponentially, which is the phase sampled by our 19 month panel.

Let g_t denote the instantaneous log cost decline t years after the last observation, and let H be the half life of the excess rate. We use

$$\begin{aligned} g_t &= g_H + (g_0 - g_H)2^{-t/H}, \\ F_T &= \exp\left\{ g_H T + (g_0 - g_H)\frac{H}{\ln 2}\left(1 - 2^{-T/H}\right)\right\}. \end{aligned} \tag{C15}$$

Here F_T is initial task cost divided by task cost after T years. We set $H = 2$ years for the central path and its confidence interval endpoints. The central path gives annual cost declines of 75, 66.7, 59.2, 52.9, and 47.9 percent in years one through five. The instantaneous annual decline falls from about 79.1 percent initially to 45.8 percent at year five and converges to 33.5 percent.

Our choice $H = 2$ is a scenario parameter, not an estimate from the AI series. Genome sequencing costs beat the hardware trend by roughly 100 fold from 2008 to 2012, and the excess in the NHGRI series subsequently faded on about a two year half life (Wetterstrand, 2023). Processor price declines in the 1990s faded more slowly (Flamm, 2018; Nordhaus, 2007). Frontier training compute grew roughly 300 to 400 percent per year in the recent scaling era (Sevilla et al., 2022), but power, fabrication, data, and capital constraints become material by 2030 (Sevilla et al., 2024). A two year half life removes three quarters of the excess rate within four years.

What Changes in the Counterfactual

An unprimed variable is a baseline level and a prime denotes its counterfactual level. For every positive variable x, its hat is

$$\hat{x} \equiv \frac{x'}{x}. \tag{C16}$$

Thus $\hat{x} = 1$ means no change and $\hat{x} = 2$ means that x doubles. The model's human and AI technique indexes in sector s are T_{Ls} and T_{Ms}. The counterfactual changes AI technique by $\hat{T}_{Ms}$, leaves human technique unchanged, and fixes each sector's capital stock and the supply of installed AI services at $\hat{M} = 1$.

The technology shocks come from the same price cost regression in Section 7. Let $p \in \{\text{low}, \text{central}, \text{high}\}$ index the point estimate and the two endpoints of its 95 percent confidence interval. Each path has an initial log decline rate g_{0p}. The hardware rate g_H and the excess half life H are common across paths. Let $F_{p,h}$ denote the cumulative cost improvement factor defined by equation (C15).

Let $b \in \{\text{equal}, \text{exposure}\}$ index the two patterns of sectoral progress described below, and let

e_s^b be the factor that multiplies the measured decline in log AI costs under pattern b, so sector s's log AI cost falls by e_s^b times the measured decline. Scaling every task's decline this way is the conservative reading: a mixture in which only a fraction of tasks follows the whole path, with the same average log decline, gives a larger cost saving by convexity. The technique change over h years is the product of the annual changes in equation (41),

$$\hat{T}_{Ms}(p,h,b) = \prod_{t=1}^{h} \left(\frac{F_{p,t}}{F_{p,t-1}} \right)^{\psi e_s^b} = F_{p,h}^{\psi e_s^b}. \tag{C17}$$

The exponent ψ converts the task cost improvement into the technique index used by the model.

Under jagged adoption, each sector uses its measured exposure e_s. Under balanced adoption, all sectors use the value added weighted mean:

$$e_s^{\text{exposure}} = e_s, \qquad e_s^{\text{equal}} = \sum_k v_k e_k. \tag{C18}$$

Both patterns have the same value added weighted mean log technique change. These value added weights are the weights in Theorem 3, so the two patterns are matched on the average the theorem uses. The initial log decline rates are 1.45, 1.57, and 1.68 for the low, central, and high paths. All use $g_H = 0.41$ and $H = 2$. We simulate each path at horizons $h \in \{1,2,3,4,5\}$ under both patterns, giving 30 mobile equilibria and the same number with fixed sector employment. The main figures use the central path.

For a given cost path, horizon, and exposure profile, the unknowns are the S sector wage changes $\{\hat{w}_s\}$ and the common AI rental change $\hat{r}$. The wage w_s is pay per unit of effective labor, and r is the price of installed AI services. We express all hats in final good units, with $\hat{P} = 1$. Fixed capital implies $\hat{R}_s = \hat{I}$ wherever $b_s > 0$, so capital rentals do not add independent unknowns. This normalization changes no real outcome.

C.5 Tasks and Sector Prices

Begin with the term that combines human and AI technique:

$$B_s \equiv T_{Ls} + T_{Ms} \left(\frac{w_s}{r} \right)^{\alpha_s \psi}. \tag{C19}$$

The human part of B_s has baseline share π_{Ls}, and the AI part has baseline share π_{Ms}. Because only AI technique changes, the hat of this term is

$$\hat{B}_s = \pi_{Ls} + \pi_{Ms} \hat{T}_{Ms} \left(\frac{\hat{w}_s}{\hat{r}} \right)^{\alpha_s \psi}. \tag{C20}$$

This equation combines the direct improvement in AI technique with the equilibrium change in the wage relative to the AI rental.

The new task shares follow immediately:

$$\pi'_{Ms} = \frac{\pi_{Ms}\hat{T}_{Ms}(\hat{w}_s/\hat{r})^{\alpha_s\psi}}{\hat{B}_s}, \qquad \pi'_{Ls} = \frac{\pi_{Ls}}{\hat{B}_s}. \tag{C21}$$

The two shares add to one. The second expression also implies $\hat{\pi}_{Ls} = 1/\hat{B}_s$: workers' task share falls exactly when the task productivity term rises.

Sector prices satisfy

$$\log\hat{P}_s = b_s\log\hat{I} + a_s\log\hat{w}_s - \frac{\theta_s}{\psi}\log\hat{B}_s + \sum_k \Gamma_{ks}\log\hat{P}_k. \tag{C22}$$

The final expenditure bundle is the numeraire, so

$$1 = \hat{P} = \prod_s \hat{P}_s^{\beta_s}. \tag{C23}$$

Higher wages and capital rentals raise direct costs. Better AI technique lowers them through $\hat{B}_s$, and supplier prices transmit cost changes through intermediate purchases. Time invariant cost constants cancel in hats.

C.6 Worker Sorting and Group Wages

Let W_g denote average earnings in group g. The Roy model combines the sector wage changes as follows:

$$\hat{W}_g = \left(\sum_s \mu_{gs}\hat{w}_s^{\kappa_g}\right)^{1/\kappa_g}. \tag{C24}$$

Equation (C24) combines the sector wage changes into group g's average earnings. Sectors with larger baseline weights matter more, although the group can respond by changing where it works. The new sorting shares are

$$\mu'_{gs} = \frac{\mu_{gs}\hat{w}_s^{\kappa_g}}{\sum_k \mu_{gk}\hat{w}_k^{\kappa_g}} = \frac{\mu_{gs}\hat{w}_s^{\kappa_g}}{\hat{W}_g^{\kappa_g}}. \tag{C25}$$

For every group, these new shares remain positive and add to one. For group g, a sector gains workers from that group if its wage rises by more than $\hat{W}_g$.

C.7 Income and Market Clearing

Total income comprises worker earnings, AI rental income, and capital income. With fixed AI services and sector capital stocks, its change satisfies $\hat{I} = L_0\sum_g \omega_g\hat{W}_g + M_0\hat{r} + c_K\hat{I}$. Moving the capital term to the left gives

$$\hat{I} = \frac{M_0 \hat{r} + L_0 \sum_g \omega_g \hat{W}_g}{1 - c_K}. \tag{C26}$$

The denominator is the baseline share paid to workers and AI. Each active sector capital rental changes in proportion to total income.

The worker wage bill in each sector must equal payments to the tasks done by its workers:

$$L_0 \sum_g \omega_g \hat{W}_g \mu'_{gs} = \hat{I} \theta_s v_s \pi'_{Ls} \qquad \text{for every } s. \tag{C27}$$

The left side allocates group wage bills across sectors. The right side takes the worker share of the sector's task bundle payments; capital payments are separate.

The aggregate price normalization is

$$1 = \hat{P} = \hat{I}^{c_K} \prod_s \hat{c}_s^{\theta_s v_s}, \qquad \hat{c}_s = \hat{w}_s \hat{B}_s^{-1/(\alpha_s \psi)}. \tag{C28}$$

The capital and task exponents sum to one. Equations (C20) to (C28) determine sector wage hats and the AI rental hat. Task shares, sorting, earnings, income, and capital rentals then follow.

AI market clearing follows by summing the sector labor equations and using the income identity:

$$M_0 \hat{r} = \hat{I} \sum_s \theta_s v_s \pi'_{Ms}. \tag{C29}$$

The two sides are AI rental income and the sum of AI payments across sectors, both relative to baseline income. This equality checks the solution and adds no independent equation.

The fixed sector benchmark. We hold each group's effective labor in each sector at its baseline level. Task assignment still responds to wages and AI costs, and sector capital stocks remain fixed. We use the same final good normalization, $\hat{P} = 1$, as in the mobile equilibrium. Group wage changes are arithmetic means of sector wage changes:

$$\hat{W}_g = \sum_s \mu_{gs} \hat{w}_s.$$

The sector labor equations become

$$L_0 \sum_g \omega_g \mu_{gs} \hat{w}_s = \hat{I} \theta_s v_s \pi'_{Ls} \qquad \text{for every } s.$$

We combine them with the task share equations, the income equation (C26), and the price normalization (C28). The baseline identity (C2) implies that every hat equals one when there is no technology change. AI market clearing again follows from the other equations.

C.8 Value Added, Real Wages, and the Worker Income Share

Since $I = Y$, real value added changes by

$$\hat{Y} = \hat{I}. \tag{C30}$$

The real wage change for group g is $\hat{W}_g$, and the real wage change in sector s is $\hat{w}_s$.

Let $\bar{L}_g$ be the fixed population of group g, and let $E_L \equiv \sum_g \bar{L}_g W_g$ denote total worker earnings. Because ω_g is group g's baseline share of those earnings,

$$\hat{E}_L = \sum_g \omega_g \hat{W}_g. \tag{C31}$$

Group populations are fixed, so $\hat{E}_L$ is also the change in the average real wage, using baseline wage bill shares as weights. This is the aggregate real wage series in Figure 7.

Workers' share of value added is

$$s'_L = \frac{L_0 \hat{E}_L}{\hat{I}} = \sum_s \theta_s v_s \pi'_{Ls}. \tag{C32}$$

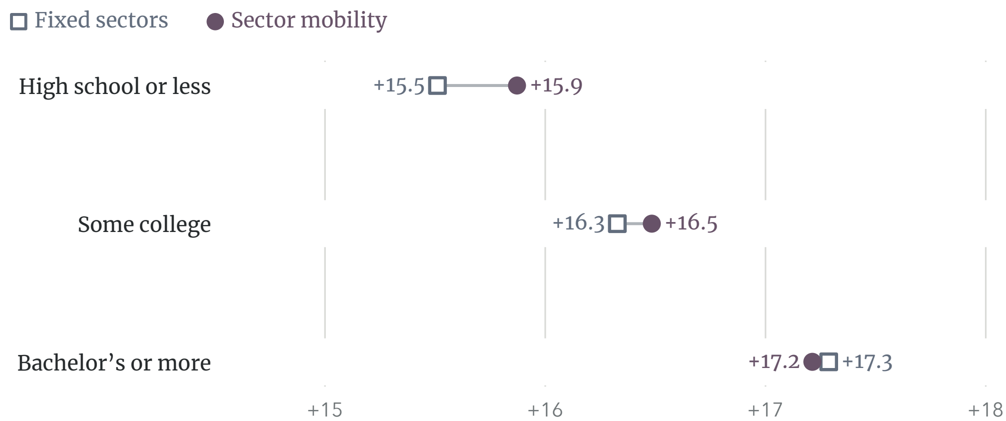
Its baseline value is $s_L = L_0$, and $\hat{s}_L = s'_L / s_L$. The worker and AI income shares sum to $1 - c_K$. Figure 7 plots $\hat{s}_L$ alongside the average real wage change $\hat{E}_L$.

Finally, the change in the wage ratio between any two groups g and g' is

$$\frac{W'_g / W'_{g'}}{W_g / W_{g'}} = \frac{\hat{W}_g}{\hat{W}_{g'}}. \tag{C33}$$

The figure below instead reports the underlying education group real wage gains, making the worker comparisons directly visible.

Figure C5. Five Year Real Wage Gains under Balanced Adoption



Notes: The figure reports five year average real wage gains for three education groups under balanced adoption. Open points show fixed sectors; solid points show sector mobility at $\kappa = 1.5$. Each number is the percentage increase relative to the wage today. In both cases, workers may switch tasks within an industry and the stock of AI capacity is held fixed. The source is our calculation.

C.9 Sufficient Statistics for Real Wage Changes with Capital

Let $N = \sum_g \bar{L}_g$ be the fixed worker population and $E = E_L/N$ average earnings, so $s_L = NE/I$. Hold human technology, input shares, final expenditure shares, and sector capital stocks fixed. If all sector wages move proportionally, the real wage change is

$$\Delta \ln w = -\frac{1}{\psi} \sum_s \theta_s \lambda_s \, \Delta \ln \pi_{Ls} + c_K \Delta \ln s_L. \tag{C34}$$

Here w can be any sector wage, since their proportional changes coincide. The first term is positive when workers perform a smaller share of tasks. The second is negative when workers' share of income falls and $c_K > 0$.

To derive the formula and its extension to different sector wage changes, use the worker assignment share in (C6) to write

$$c_s = w_s \left(\frac{\pi_{Ls}}{T_{Ls}} \right)^{1/(\alpha_s \psi)}, \qquad \Delta \ln c_s = \Delta \ln w_s + \frac{1}{\alpha_s \psi} \Delta \ln \pi_{Ls}.$$

The second equality uses fixed T_{Ls} and α_s. Inserting it into the sector cost function gives

$$\Delta \ln P_s = b_s \Delta \ln R_s + a_s \Delta \ln w_s + \frac{\theta_s}{\psi} \Delta \ln \pi_{Ls} + \sum_k \Gamma_{ks} \Delta \ln P_k.$$

Multiply these equations by the Domar weights and sum. The identity $\boldsymbol{\beta} = (\mathrm{Id} - \Gamma)\boldsymbol{\lambda}$ cancels intermediate purchases and leaves the final expenditure price change. Since $\Delta \ln R_s = \Delta \ln I$ for every sector with capital payments, $\lambda_s a_s = \theta_s v_s$, and $\sum_s \lambda_s b_s = c_K$, we obtain

$$0 = \Delta \ln P = c_K \Delta \ln I + \sum_s \theta_s v_s \Delta \ln w_s + \frac{1}{\psi} \sum_s \theta_s \lambda_s \Delta \ln \pi_{Ls}. \tag{C35}$$

The income share identity gives $\Delta \ln I = \Delta \ln E - \Delta \ln s_L$, because N is fixed. Substituting and subtracting the price change from earnings growth yields the exact formula

$$\begin{aligned} \Delta \ln E &= -\frac{1}{\psi} \sum_s \theta_s \lambda_s \Delta \ln \pi_{Ls} + c_K \Delta \ln s_L + \mathcal{R}, \\ \mathcal{R} &\equiv (1 - c_K) \Delta \ln E - \sum_s \theta_s v_s \Delta \ln w_s. \end{aligned} \tag{C36}$$

This identity applies to both mobile and fixed sector workers. The term $\mathcal{R}$ accounts for differences between average earnings growth and the sector wage changes weighted by task payments; it cannot generally be inferred from task and income shares alone.

If all sector wages change by the same factor, group earnings and average earnings change by that factor as well. Since $\sum_s \theta_s v_s = 1 - c_K$, we then have $\mathcal{R} = 0$, proving (C34). Setting $\theta_s = 1$ in every sector also gives $c_K = 0$ and recovers the common wage input output formula (30). With fixed capital, declining worker income shares reduce how much of the task related cost saving appears as real wage growth.

C.10 Numerical Solution and Checks

We solve the equilibrium in two ways. The primary calculation uses log wage changes relative to the AI rental and minimizes

$$\mathcal{V} = (1 - c_K) \log\left(\frac{\hat{I}}{\hat{r}}\right) - \sum_s \theta_s v_s \log\left(\frac{\hat{c}_s}{\hat{r}}\right). \tag{C37}$$

Its derivative with respect to each log relative wage is the sector's labor market imbalance divided by counterfactual income. The first term is convex in log wages, while each negative task cost term is strictly convex when both techniques have positive shares. Positive baseline worker and AI payments make the objective grow without bound when log wages move to extremes. The minimum is therefore unique. We use BFGS, taking the preceding horizon's solution as the next starting point.

A separately coded least squares calculation checks the S sector wage hats and common AI rental hat in final good units. It evaluates $S + 2$ residuals, sector labor clearing, aggregate price normalization, and AI clearing, for $S + 1$ unknowns; AI clearing is redundant. Across 60 mobile and fixed sector equilibria, the largest market imbalance is 3.9×10^{-13} and the largest equation error in the independent calculation is 8.8×10^{-12}. A no shock calculation returns one for every hat. We also check derivatives, changes in sector ordering, and small and large technology shocks.

The calculation uses exact proportional changes as in (Dekle, Eaton and Kortum, 2008; Costinot and Rodríguez-Clare, 2014).

C.11 An Alternative Estimate of Current AI Use

We run the same analysis using Microsoft's *Working with AI* study to extend the survey to the remaining tasks (Tomlinson et al., 2025). Microsoft measures the share of Copilot conversations devoted to AI performing each task. We match its task descriptions to O*NET. The 332 descriptions match, and the average score for tasks outside the survey is 0.25 times the surveyed score. The aggregate AI task share is 6.8 percent. The task and employment shares, estimated survey shares, and calibration method are the same as in the Anthropic analysis. We calibrate capital shares and worker allocation using these AI shares.

Figure C6 repeats the series in Figure 7 under balanced and jagged adoption using the Microsoft starting share. For the Microsoft calibration, we infer the task cost shares from the same BEA compensation data and adjust worker allocations to the implied wage bills. The AI cost path and the remaining calibration rules are unchanged. After five years, the main calibration raises real wages by about 16.8 percent under equal progress and 18.5 percent when adoption is faster in more AI exposed industries. The corresponding gains under the 6.8 percent calibration are about 6.7 and 7.7 percent. The Microsoft baseline reduces the five year wage gain to about half of the Anthropic result in both panels. Each subsequent AI improvement applies to a smaller share of the economy from the outset.

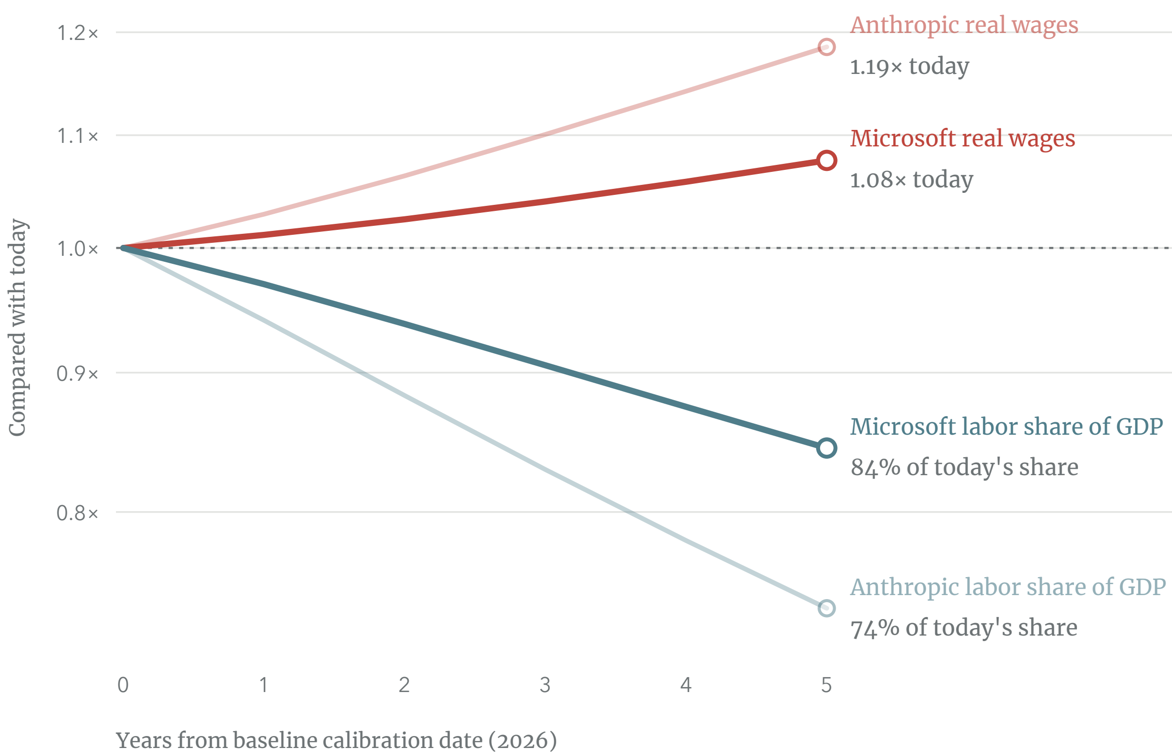
C.12 External Validation of the Model

Following Adão, Costinot and Donaldson (2025), we test whether the model correctly predicts the causal impact of artificial intelligence on wage changes across sectors. The model need not explain every observed wage change. Wages also respond to all other shocks occurring during the same period. These shocks may be large and may differ across sectors; the test requires only that they not be systematically related to the instrument defined below.

We measure wage changes between May 2022 and May 2025 using the Occupational Employment and Wage Statistics produced by the United States Bureau of Labor Statistics. For each of 61 sectors, we first calculate the change in the natural logarithm of mean hourly pay for every occupation reported in both years. We then average these changes using the sector's May 2022 occupation shares. Holding these shares fixed prevents changes in a sector's occupation mix from being counted as wage changes. Multiplying a change in the natural logarithm by 100 gives approximately the corresponding percentage change. The wage data are measured in current dollars and are not adjusted for changes in prices.

Let Δw_s^D denote the observed wage change in sector s. We subtract the average across sectors from this and every other variable used in the test. Following Adão, Costinot and Donaldson

Figure C6. Real Wages and Worker Income Shares under Two Estimates of Current AI Use



Notes: For each measure, the line with the smaller change uses Microsoft's estimate of current AI use, and the other line uses the Anthropic estimate. Each line above one shows the real wage, and each line below one shows the worker income share. Under balanced adoption, which the figure does not show, real wages after five years would rise by 6.7 percent with Microsoft's estimate and 16.8 percent with the Anthropic estimate, and workers' share of income would fall by 15 and 26.8 percent, respectively. With faster progress in more AI exposed industries, as in the figure, real wages rise by 7.7 and 18.5 percent with Microsoft's estimate and the Anthropic estimate, respectively, and worker income shares fall by 15.6 and 26.3 percent. Every line is relative to its value today and uses the same log scale. The figure shows jagged adoption. It uses the paper's baseline projection for AI progress. No other assumption changes. The sources are the worker survey, Anthropic Economic Index, Microsoft's record of AI use, O*NET task descriptions, and US industry and worker data.

(2025), we write the observed change as

$$\Delta w_s^D = \Delta w_s^* + \Delta \eta_s^*, \tag{C38}$$

where Δw_s^* is the true causal impact of artificial intelligence and $\Delta \eta_s^*$ is the causal impact of all other shocks. Neither term is observed separately. The model predicts a causal impact, denoted by Δw_s. The null hypothesis is $\Delta w_s = \Delta w_s^*$ for every sector, so under the null $\Delta w_s^D - \Delta w_s = \Delta \eta_s^*$.

We construct Δw_s from the paper's July 2026 sector assignment shares, which are based on Anthropic usage data and imply that about 14.2 percent of tasks are assigned to artificial intelligence. To align the model with the May 2025 endpoint of the wage data, we set the aggregate May 2025 assignment share equal to 0.087/0.217, or about 40 percent, of the July 2026 share and solve the model again. The two numbers are the national shares of businesses reporting current use of artificial intelligence in the relevant Business Trends and Outlook Survey waves. The survey records whether a business uses artificial intelligence at all, not the share of tasks assigned to it, so the scaling is approximate; we report the test without it below. For each sector, Δw_s is the model's predicted wage change between a near zero assignment share and the May 2025 assignment share, after subtracting the average predicted change across sectors. This is the definition of the causal impact used by Adão, Costinot and Donaldson (2025, footnote 11): the effect of the change in artificial intelligence use is measured in the economy observed at the end of the period, with all other shocks held at their realized values, rather than in the economy of 2022. The two definitions coincide only when the model is additively separable in the two kinds of shocks, which ours is not. Under the definition used here, the term for other shocks in equation (C38) is the wage effect of all shocks other than artificial intelligence with the assignment share held at its 2022 level, and the identifying condition stated below refers to that term.

The wage changes used in this exercise play no role in choosing the model's parameters. We also hold those parameters fixed when measuring uncertainty. We calculate the prediction under two assumptions about worker movement. In the first, workers remain in their original sectors, represented by $\kappa \downarrow 1$. In the second, workers can move across sectors and $\kappa = 1.5$, following Galle, Rodríguez Clare, and Yi (2023). Nothing else changes between the two cases.

The test uses an instrument, meaning an outside measure of which sectors should be more affected by artificial intelligence. We construct it from each sector's May 2022 occupation mix and the exposure scores of Eloundou, Manning, Mishkin, and Rock (2024). These scores measure how strongly language models could affect the tasks performed in each occupation.

Let ℓ_{so} denote occupation o's share of employment in sector s, and let h_o denote its exposure score. A tilde over ℓ_{so} means that we subtract occupation o's average employment share across sectors. A tilde over h_o means that we subtract the average exposure score across occupations. The instrument is

$$z_s = \sum_o \widetilde{\ell}_{so} \widetilde{h}_o. \tag{C39}$$

A sector has a higher value of z_s when it initially employs more workers in occupations whose tasks are more exposed to language models. The calculation uses 786 distinct occupation level exposure terms.

The identifying condition required by the test is

$$\mathbb{E}_t \left[\frac{1}{S} \sum_s z_s \Delta \eta_s^* \right] = 0, \qquad S = 61. \tag{C40}$$

Here $\mathbb{E}_t$ denotes expectation conditional on the initial state of the economy. In words, sectors with higher values of the instrument must not experience systematically different wage changes for reasons other than artificial intelligence. This condition does not require the other shocks to be small or absent. It requires only that their wage effects have no systematic relationship with the instrument when averaged across sectors.

Adão, Costinot and Donaldson (2025) measure goodness of fit using

$$\widehat{\beta}_z = \frac{1}{S} \sum_s z_s \left(\Delta w_s^D - \Delta w_s\right). \tag{C41}$$

This measure is the covariance of the instrument with the difference between observed and predicted wage changes, calculated as their average product because the variables have been demeaned. It is positive when sectors with higher values of the instrument tend to have wage changes above the model prediction and negative when they tend to have wage changes below it. Under the null hypothesis and the identifying condition, $\mathbb{E}_t[\widehat{\beta}_z] = 0$.

Following equation (6) of Adão, Costinot and Donaldson (2025), we calculate the estimated standard error as

$$\widehat{\sigma}(\widehat{\beta}_z) = \left\{ \sum_o \widetilde{h}_o^2 \left[\frac{1}{S} \sum_s \widetilde{\ell}_{so} \left(\Delta w_s^D - \Delta w_s\right) \right]^2 \right\}^{1/2}. \tag{C42}$$

The standard error measures how much the goodness of fit statistic would vary across realizations of the exposure scores and other shocks. Following their assumptions for inference, we treat the occupation level exposure scores as independent draws from a common distribution, conditional on the initial state of the economy, and require that no small group of occupations dominate the instrument. It allows wage changes across sectors to be related. We use a two sided normal approximation to calculate p values. Two further checks relax the independence assumption. First, we allow the exposure scores to be correlated within occupation groups defined by three digit occupation codes, summing the occupation terms of equation (C42) within each group before squaring; the p values become 0.7 when workers can move across sectors and 0.48 when they remain in their original sectors. Second, we randomly reassign the demeaned exposure scores across occupation groups of equal size and compare the statistic with its permutation distribution, which gives p values of 0.76 and 0.72. No single occupation dominates the instrument: the largest occupation accounts for 10.5 percent of the sum of squared occupation loadings.

We also report the instrumental variables (IV) coefficient, which expresses the same comparison as a ratio:

$$\begin{aligned} \widehat{\beta}^{IV} &= \frac{S^{-1} \sum_s z_s \Delta w_s^D}{S^{-1} \sum_s z_s \Delta w_s}, \\ \widehat{\beta}_z &= \left(S^{-1} \sum_s z_s \Delta w_s \right) (\widehat{\beta}^{IV} - 1). \end{aligned} \tag{C43}$$

The numerator is the covariance of the instrument with observed wage changes, and the denominator is its covariance with the model prediction. Thus, $\widehat{\beta}^{IV} = 1$ means that these two covariances are equal. It does not require observed and predicted wage changes to coincide in each sector. The second equality links the ratio's deviation from one to $\widehat{\beta}_z$ whenever the denominator is nonzero. We construct a 95 percent Anderson Rubin confidence set by applying the same test to $\Delta w_s^D - \theta \Delta w_s$ over a closely spaced grid of candidate values of θ and retaining those not rejected at the 5 percent significance level. At $\theta = 1$, this is exactly the test based on $\widehat{\beta}_z$.

The results do not reject the model's causal prediction at the 5 percent significance level under either assumption about worker movement. When workers remain in their original sectors, the average product of the instrument and the observed wage change is -6.63×10^{-4}, while the corresponding model value is -8.33×10^{-4}. The observed value minus the model value is 1.7×10^{-4}, with an estimated standard error of 1.66×10^{-4}. If the model prediction and the condition on the instrument were both correct, a discrepancy at least this far from zero would occur approximately 30.5 percent of the time. The ratio estimate is 0.8, and its 95 percent confidence set is $[0.43, 1.21]$.

When workers can move across sectors and $\kappa = 1.5$, the corresponding model value is -5.77×10^{-4}. The observed value minus the model value is -8.6×10^{-5}, with an estimated standard error of 1.57×10^{-4}. A discrepancy at least this far from zero would occur approximately 58.4 percent of the time if the model prediction and the condition on the instrument were both correct. The ratio estimate is 1.15, and its 95 percent confidence set is $[0.62, 1.74]$.

Both confidence sets contain one, and neither p value is below 5 percent. We therefore do not reject the model's causal prediction under either assumption about worker movement. The scaling of the assignment share matters little: using the July 2026 share without rescaling, with workers able to move across sectors, gives an IV coefficient of 0.79 with a 95 percent confidence set of $[0.44, 1.17]$ and a p value of 0.26 for the goodness of fit statistic.

The result with workers remaining in their original sectors may be especially relevant to the three years covered by the wage data, since movement across sectors takes time. The result for $\kappa = 1.5$ uses the same mobility parameter as the paper's five year exercises.

The exercise tests differences in wage growth across sectors, not the average wage change for the economy, employment, or welfare. Its interpretation also depends on the approximate scaling between the two Business Trends and Outlook Survey waves, which use different wording. The wage data are measured in current dollars, whereas the model measures pay after accounting for differences in worker productivity. The published wage figures are based on a sample, some sector occupation cells are not reported, and the mix of workers within an occupation may change over time. Subject to these limits and the required condition on the instrument, the test does not detect a discrepancy large enough to reject the model's predicted causal impact on relative wages at the 5 percent significance level.

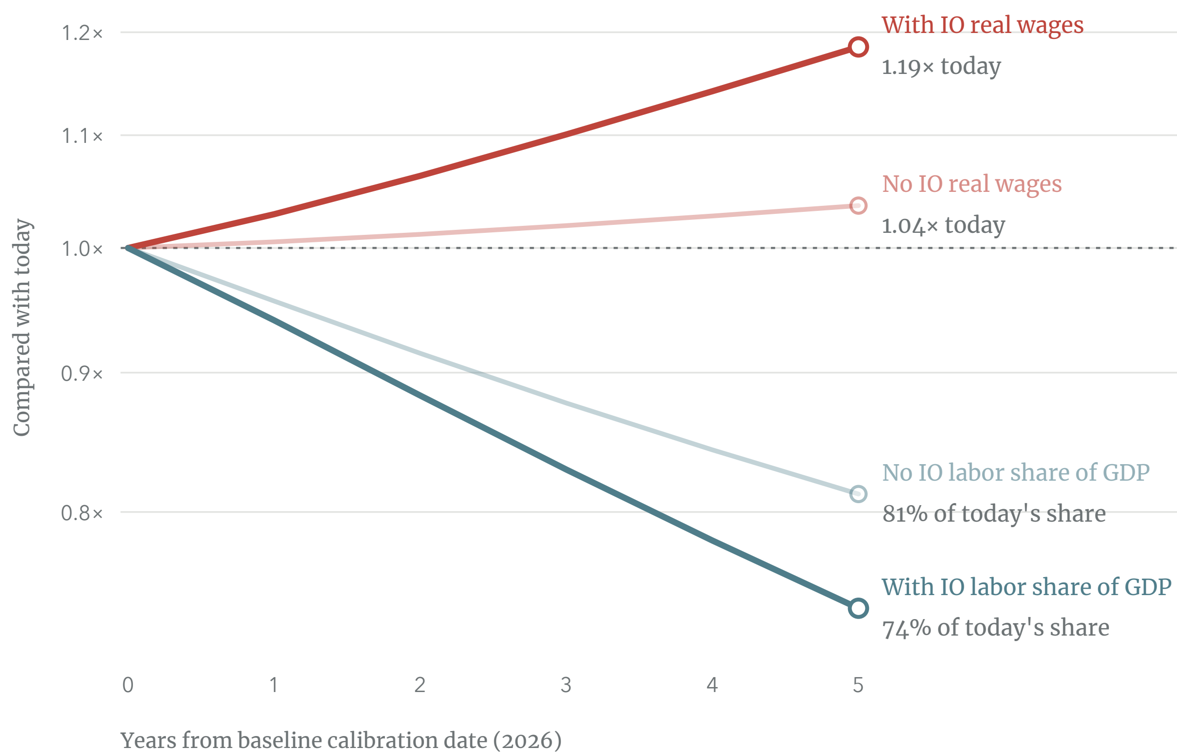
C.13 The Role of Input Output Networks

To measure the contribution of intermediate inputs, we compare the calibrated economy with one in which $\Gamma = 0$ and $\alpha_s = 1$ in every sector. The comparison uses Domar weights $\lambda_s = v_s$ and keeps θ_s, c_K, the value added weights, initial task shares, worker shares, installed AI services, and AI progress scenarios fixed. It therefore removes intermediate purchases while retaining the same capital shares.

Figure C7 reports average real wage growth and the worker income share with and without intermediate inputs. After five years of balanced adoption, real wages rise by 16.8 percent with the network and 2 percent without it. Under jagged adoption, the gains are 18.5 and 3.6 percent. A cost saving in one sector lowers the prices paid by sectors that buy its output, transmitting gains through intermediate purchases. The Domar weights sum to 1.78 rather than one. The wage response also affects task assignment less when the primary input accounts for a smaller share of task cost.

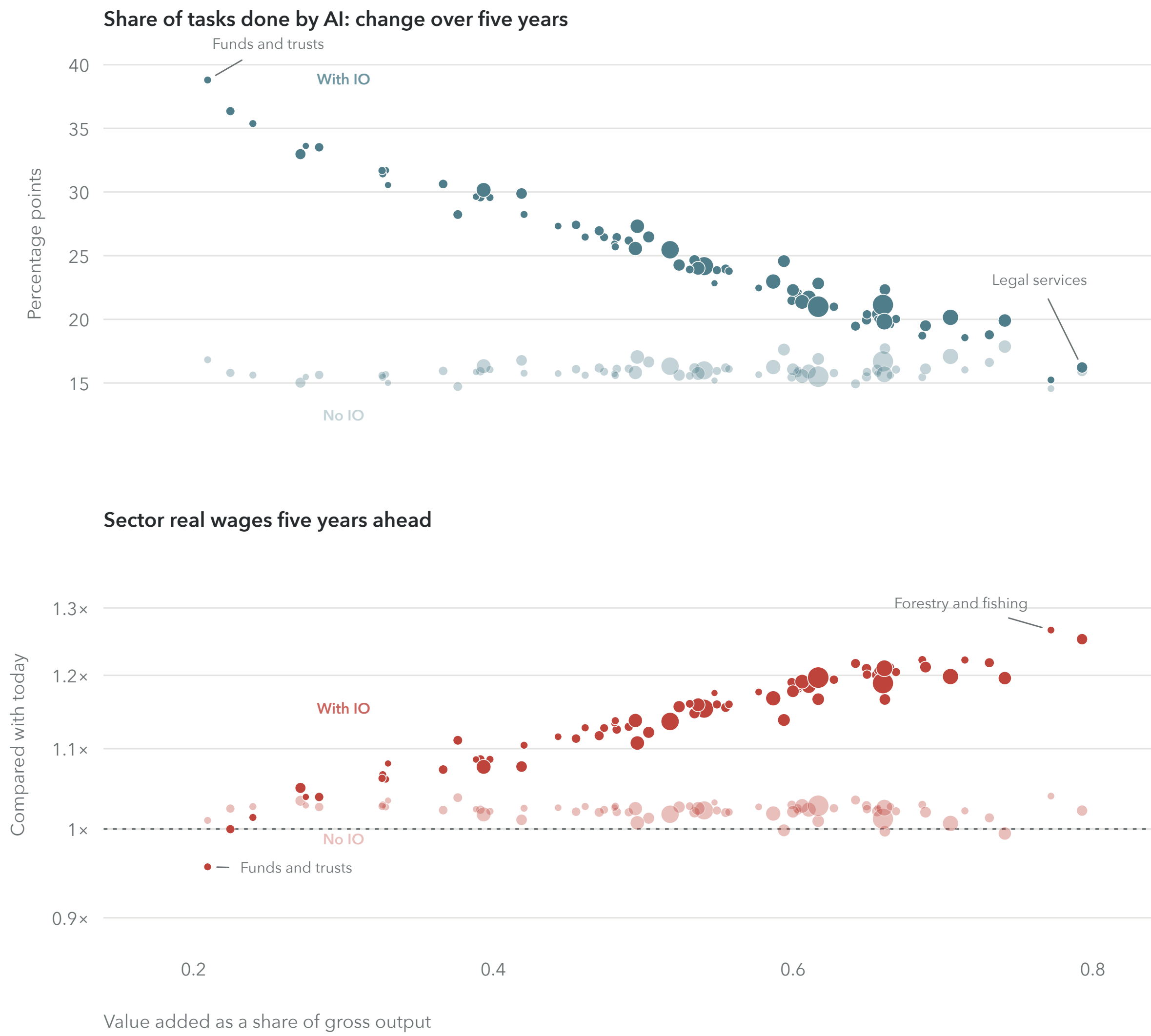
In Figure C8, the five year increase in the AI task share under equal progress ranges from 14 to 39 percentage points in the baseline, compared with 15 to 18 points without intermediate inputs. The AI task share rises by 39 points in funds, trusts, and other financial vehicles, where value added is a small share of gross output, and by 16 points in legal services, where it is a large share. With limited worker mobility, sectors with larger shifts toward AI have smaller real wage gains (lower panel).

Figure C7. Real Wages and Worker Income Shares with and without Input Output Linkages



Notes: Each line is relative to its value today and equals one at year zero, on a log scale. Aggregate real wages are the mean of the three education groups' real wages, weighted by baseline wage bills. Each teal line is workers' share of total value added. For each measure, the line with the larger change shows the baseline economy with the BEA input output network, and the other line shows the economy with $\Gamma = 0$ and $\alpha_s = 1$ in every sector. The figure shows faster progress in more AI exposed industries. Both simulations use the central cost path in Figure 6, hold the stock of AI capacity fixed, set the change in each sector's AI technology as in equation (41), and use the equilibrium in which workers move across sectors at the calibrated $\kappa = 1.5$.

Figure C8. Sector Task Shares and Real Wages with and without Input Output Linkages



Notes: One point per sector, 71 in all, shown against the sector's value added as a share of its gross output; the size of each point is proportional to the sector's share of value added. The upper panel shows the change over five years in the share of tasks done by AI, in percentage points. The lower panel shows the sector real wage after five years relative to its value today, on a log scale. One set of points shows the baseline economy with the BEA input output network, and the other shows the economy with $\Gamma = 0$ and $\alpha_s = 1$ in every sector. Each panel uses the same progress in every industry, the central cost path in Figure 6, the stock of AI capacity held fixed, the change in each sector's AI technology as in equation (41), and the equilibrium in which workers move across sectors at the calibrated $\kappa = 1.5$.

D Measurement Details and Robustness

D.1 Data Construction

We construct a panel of 32 system configurations and 484 software engineering tasks, observed over releases from 2024 through 2026. The task set originates with Jimenez et al. (2024); we retain the tasks that every configuration in the panel attempts. A task is a real defect or feature request, filed against one of 12 widely used open source software projects (repositories) and since resolved by a human developer. It is posed as the project received it: a written statement of what is wrong, together with the project's code frozen just before the human repair. Success is mechanical rather than judged: the project's own tests, written when the human resolved the issue and withheld from the AI system, must pass, and tests that passed beforehand must still pass. There is no partial credit.

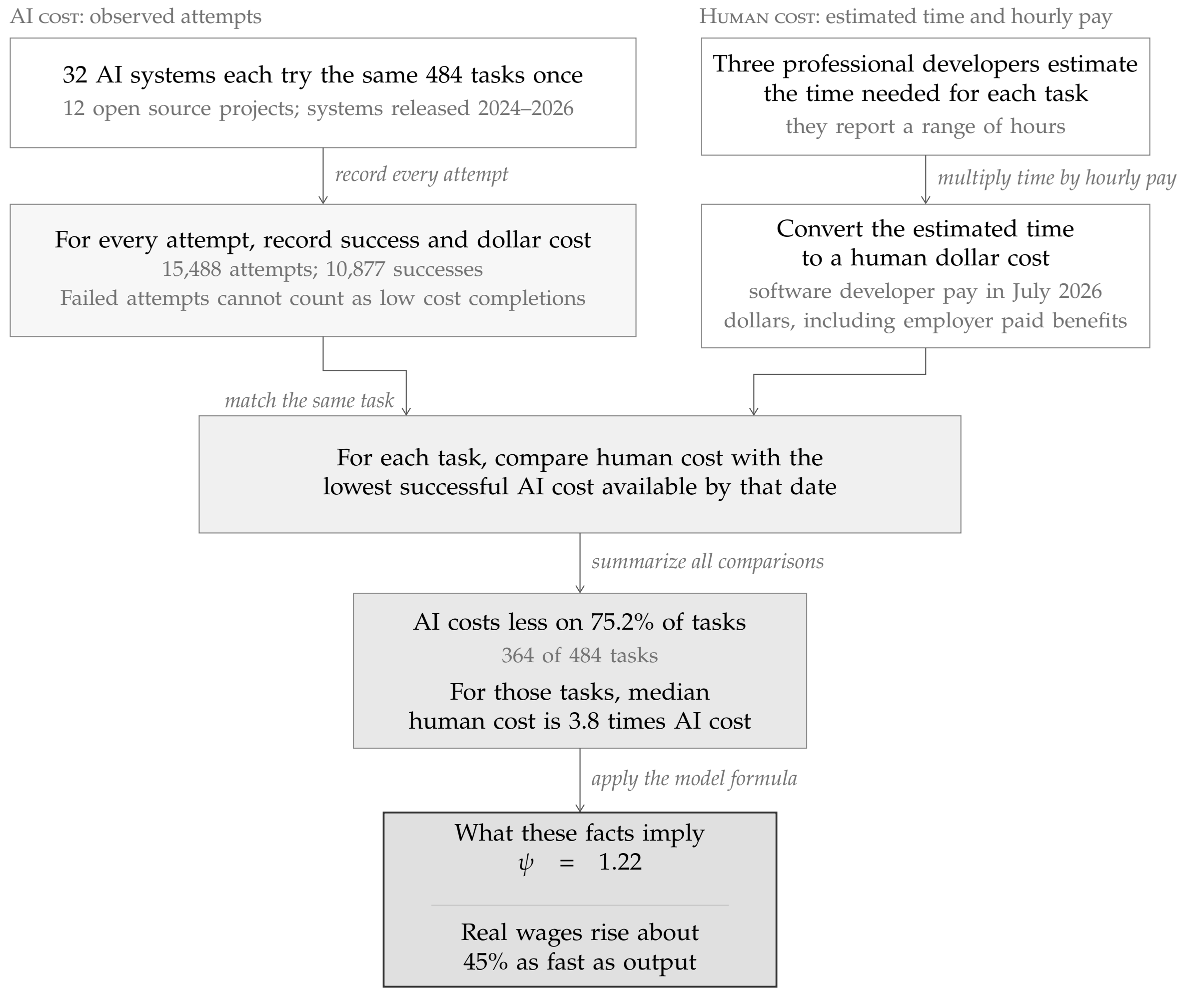
Figure D1 shows how the cost records determine the model's wage response. We compare the lowest cost of a successful AI attempt with the estimated human cost for the same task. Failed attempts remain in the data but cannot count as low cost completions.

Two facts determine ψ. AI is cheaper on 75.2 percent of the tasks. On those tasks, the median human cost is 3.8 times the AI cost. Together they imply $\psi = 1.22$, so real wages rise about 45 percent as fast as output. The estimate comes entirely from task costs, not from fitting wages or output.

The unit on the AI side is a system configuration: a particular model together with the working arrangement it is given, namely which software tools it may operate and how much deliberation it is permitted. Epoch AI, an independent research organization, runs each configuration on every task exactly once, with no retries, and supplies the outcomes and recorded token use (Epoch AI, 2026*a*). Neither we nor the model vendors conduct the measurement. The resulting 15,488 configuration task observations cover the 12 repositories.

We measure costs on both sides of each potential assignment. For AI, cost is the recorded token use of a demonstrated single attempt completion, priced at the vendor's posted rates on the date of the attempt. We multiply this billed cost by a fixed factor $r^* = 175$ that stands for the costs of deploying AI beyond tokens. It is calibrated to the shares of AI-written code reported by Microsoft (30 percent, April 2025) and Google (50 percent, November 2025; 75 percent, April 2026): for each statement we find the factor at which the share of tasks assigned to AI on the statement's date equals the reported share, and $\ln r^*$ is the unweighted mean of the three implied log factors. A share of code written is not a share of tasks completed at below human cost, so this calibration is an assumption rather than a measurement. A configuration that does not complete a task receives no finite completion cost and is not assigned that task. Assignment is measured against the cumulative system frontier, the running minimum AI cost across all configurations released to date, rather than against any single configuration.

Figure D1. How Human and AI Task Costs Determine the Wage Response



Notes: Each AI system makes one attempt at every task. A failed attempt cannot complete a task more cheaply than a worker. For each task and date, AI cost is the cheapest successful attempt available. Human cost is the estimated completion time multiplied by a software developer wage that includes benefits. Table B1 reports the remaining assumptions. Our calculation uses the record of each AI attempt and its cost from Epoch AI (Epoch AI, 2026*a*) and professional estimates of human completion times from OpenAI (2024).

For labor, cost is a professionally assessed completion time priced at an hourly wage. The assessments come from the verification exercise of OpenAI (2024), in which three experienced developers place each task in one of four time bands, the uppermost of which is open ended. We price assessed time at 3 wage bases: the mean BLS software developer wage, a loaded BLS wage, and a contractor rate (U.S. Bureau of Labor Statistics, 2024). We convert all nominal amounts to July 2026 dollars using the consumer price index for urban consumers (U.S. Bureau of Labor Statistics, 2026*a*).

The assessed times are interval censored. Our pre specified conventions use arithmetic or geometric point midpoints, or integrate uniformly in levels or in logs; the open ended upper interval is capped at either 8 or 16 hours. For the distributional conventions, we consider both comonotone and independent dependence across task time intervals. The headline measure uses the loaded BLS wage, the 16 hour cap, geometric midpoints, and the cumulative system frontier through June 2026. Table B1 summarizes these choices.

Table B1. Data Construction Summary

Component	Construction
Panel	32 system configurations by 484 tasks; 15,488 configuration task observations from 12 repositories.
AI cost	Realized compute expenditure for demonstrated single attempt completion, scaled by the factor r^*. Uncompleted tasks receive no finite completion cost.
Human cost	Professionally assessed completion time intervals multiplied by an hourly wage.
Wage bases	Mean BLS software developer wage; loaded BLS wage at 1.4 times the mean; contractor rate of $100 per hour.
Prices	Nominal amounts converted with the consumer price index for urban consumers to July 2026 US dollars.
Time conventions	Arithmetic and geometric point midpoints, uniform in levels and uniform in logs integration, and 8 or 16 hour upper caps.
Dependence	Comonotone and independent integration for distributional interval conventions.
Headline	Loaded BLS wage, 16 hour cap, geometric midpoint, and the cumulative system frontier through June 2026.

Notes: Each assessed range of hours comes from the professional assessment described in OpenAI (2024). BLS denotes the Occupational Employment and Wage Statistics data. For tasks assessed at under 1/4 hour, the midpoint in logs is $\sqrt{(1/60) \times (1/4)} \simeq 0.065$ hours; the conventions in logs also use a lower bound of 1/60 hour. Appendix D.1 provides further details on the construction and provenance.

The panel measures single attempt completion and relative cost on the observed task distribution. This assignment experiment identifies the Fréchet dispersion parameter used in the economy wide counterfactuals of Section 8.

D.2 Estimation and Inference

We use the notation of Section 7. The two estimates of ψ reported in the paper come from the two regressions of that section, equations (38) and (39), and are estimated exactly as described there.

At the endpoint date, $c_M(i)$ is the cumulative AI frontier cost described above: the lowest cost at which any evaluated system configuration released by that date completed the task successfully. The cost of a human doing the same task is $c_L(i)$. For a task the frontier completes at least once, the ratio of the two costs is $D(i) = c_L(i)/c_M(i)$. We define $\widehat{\pi}_M$ as the share of all tasks the frontier completes for less than the human cost, that is, the tasks with a successful completion and $D(i) > 1$. A task with no successful completion, or whose cheapest successful completion still costs at least as much as a human, stays in the denominator and counts as unassigned. Among the tasks assigned to AI, $\ln \widehat{m}$ is the sample median of $\ln D(i)$. Every task receives equal weight.

Evaluating the mapping in equation (37) at the endpoint share and median gives $\widehat{\psi} = 1.22$, with a bootstrap standard error of 0.08. This endpoint value lies between the two regression estimates; the counterfactuals impose the task level estimate $\psi = 1.36$.

We measure uncertainty in the endpoint estimate by resampling software projects. Each of the 1,000 draws selects 12 projects with replacement from the projects in the benchmark and keeps every task and every system attempt from a selected project together. A project selected more than once enters as that many separate copies. For each draw we rebuild the cumulative AI frontier with r^* held at its full sample value, recompute $\widehat{\pi}_M$ and $\widehat{m}$, and evaluate equation (37) again. The draws use a fixed seed set in advance, and the standard error is the standard deviation of the resulting estimates. The two regression estimates reported in the paper use the same project resampling, with 400 draws. Resampling this way treats projects as unrelated to one another and lets tasks within the same project move together, so the standard error measures how much $\widehat{\psi}$ moves when the mix of software projects in the benchmark changes. It holds fixed the systems that were evaluated, the single attempt recorded for each of them, and the costing conventions; it does not cover the uncertainty that would come from rerunning the evaluations or from choosing a different set of systems. The assessed completion times are held fixed across draws as well, so the grid of costing conventions in Appendix D.3 is the relevant sensitivity to measurement error in human costs.

D.3 Human Costs from Assessed Time

Human costs are assessed completion times priced at wages, not transacted prices. The assessments are interval censored, so any point estimate also requires a convention for resolving the intervals. We resolve them in six ways: at the arithmetic or the geometric midpoint of each interval, or by integrating over the interval uniformly in levels or in logs, each of the two integrating conventions under either comonotone or independent dependence across tasks. The open ended upper interval is capped at 8 or 16 hours, and we price assessed time at the loaded BLS wage, the mean BLS wage, or the contractor rate. Figure B1 reports the implied ψ under all eighteen combinations of convention and wage basis. The two caps deliver the same value for many of these combinations and move ψ only slightly elsewhere, so the figure marks the lower cap only where it differs.

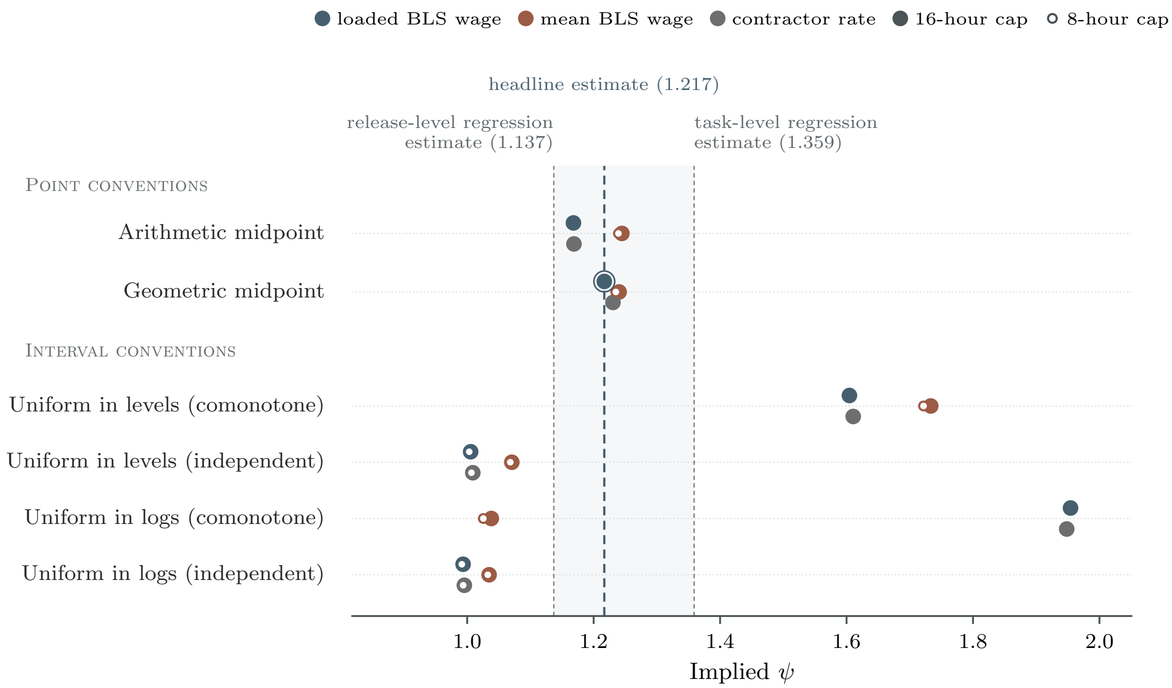


Figure B1. Implied ψ Across Wage Bases and Assessed Time Conventions

Notes: For each of the conventions for the assessed range of hours, the figure reports the estimated ψ under each wage basis, which the colors show: the BLS wage scaled up by 1.4 to cover benefits and overhead, the BLS mean wage, and the freelance rate. The two midpoint conventions use the midpoint of each range in levels or in logs; the four uniform conventions instead treat the time as uniform over the range, in levels or in logs, with perfect correlation across tasks or with independence. The midpoint conventions do not depend on the correlation across tasks. The time for tasks marked as "more than four hours" never exceeds 16 hours (solid circle) or 8 hours (small open circle); where the two coincide, only a solid circle is shown. The figure shows the range between the paper's two estimates, from 1.14 for the release-level regression to 1.36 for the task-level regression; a line shows the headline ψ of 1.22; and the point within an open circle is the baseline calibration, which uses the BLS wage scaled up by 1.4, the midpoint in logs, an upper limit of 16 hours, and the frontier through June 2026. Across these conventions the estimated ψ stays between 0.99 and 1.95, which implies real wage growth between 34 and 50 percent of the output growth rate. The source is our calculation of the implied ψ from the share and the median at the June 2026 release under each of the conventions.

The wage convention moves ψ modestly and the counterfactual outcomes very little. Under every wage basis, worker real earnings and the labor task share at the five year horizon stay close to their baseline values in both sectoral incidence patterns, so the quantitative conclusions do not hinge on how human time is priced. At $\psi = 1.22$, the balanced five year real output gain is 57 percent rather than 59.5 percent, and the gain in worker real earnings is 18.3 percent rather than 16.8 percent. Across the wage bases, time conventions, caps, and dependence assumptions in Figure B1, ψ lies in $[0.99, 1.95]$, implying real wage growth between 34 and 50 percent of the output growth rate. The assignment share ranges from 60.95 to 85.33 percent. The point conventions leave ψ inside the band traced by the paper's two regression estimates. Integrating over the intervals moves it below the band under independence and, in most cases, above the band under comonotone dependence.

The headline interval choice is conservative among the point conventions: the geometric midpoint produces the larger ψ and hence slower real wage growth relative to output. The assessed intervals themselves come from professional verification of the task set rather than from our judgment.

D.4 Occupational and Task Set Scope

The evidence comes from one occupation and a task set on which the cumulative system frontier is close to saturation. Capability is measured by single attempt completion on the observed task distribution, and the cumulative frontier through June 2026 completes approximately 75 percent of tasks at below human cost. Little of this task set therefore remains available to reveal further movement on the extensive margin.

This frontier is part of the design. The theory's post displacement objects are measured where displacement is already far along, making software engineering the natural laboratory; it is also a large, high wage occupation in the BLS data (U.S. Bureau of Labor Statistics, 2024). The leveling of assignment does not mark a technological plateau: the cost of demonstrated completion continues to fall at approximately 79 percent per year over the same releases.

The estimated value of ψ is the Fréchet dispersion parameter used across sectors in Section 8. Current sectoral assignment comes from the Epoch AI Ipsos survey, O*NET task mappings, and OEWS employment. Sectoral LLM exposure governs only the distribution of future technique progress.

D.5 Beyond the Observed Support

The data do not reveal the endpoint of comparative advantage. At the cumulative frontier, AI completes 448 of the 484 tasks and is cheaper than a human on 364 of them; the other 84 completed tasks cost more than a human, and 36 tasks are never completed. Across the panel, 5,816 completed configuration task rows cost more than a human priced at the upper end of the assessed time, and 4,611 configuration task observations do not demonstrate completion and therefore carry no finite completion cost signal. We have no relative cost measure for the tasks that are never completed, so the data cannot bound how far AI's comparative advantage ultimately extends.

This limitation leaves the theory's destination cases open. A full support power tail, a lighter unbounded tail, and a finite endpoint with a wage plateau remain consistent with what is observed (Section 3 and Theorem 1.b). Within the maintained date t calibration, none of these possibilities changes real wage growth relative to output at the measured ψ; they differ in what happens beyond the observed support. Choosing among them is a forecasting problem that the model organizes but the measurement cannot settle. The data identify how real wage growth currently compares with output growth; they do not identify the trajectory it eventually follows.

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